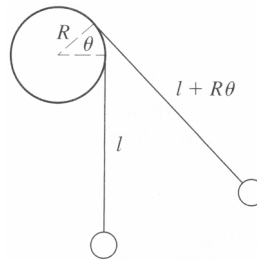


Solutions to Midterm Exam 2003

1. The figure below shows a simple pendulum system in which a cord is wrapped around a fixed cylinder. The motion of the system that results is described by the differential equation:

$$(l + R\theta)\ddot{\theta} + g \sin(\theta) + R\dot{\theta}^2 = 0,$$

where l is the length of the cord in the vertical (down) position and R is the radius of the cylinder.



- a. Write the state space equations for this system.
- b. Linearize the equation around the point $\theta = 0$, $\dot{\theta} = 0$, and show that for small values of θ the system equation reduces to an equation for a simple pendulum, that is, $\ddot{\theta} + (g/l)\theta = 0$. (25 pts)
- a. Let $x_1(t) = \theta$, $x_2(t) = \dot{\theta}$ and $y(t) = \theta$. Substituting these into the differential equation gives:

$$(l + Rx_1(t))\dot{x}_2(t) + g \sin(x_1(t)) + Rx_2(t)^2 = 0$$

$$\Rightarrow \dot{x}_2(t) = \frac{-g \sin(x_1(t)) - Rx_2(t)^2}{(l + Rx_1(t))},$$

the rate of change of the second state variable with respect to the present values of the state variables. We can determine the rate of change of the first state variable directly from its definition:

$$\dot{x}_1(t) = \dot{\theta} = x_2(t).$$

Likewise, from its definition, $y(t) = \theta = x_1(t)$.

The state space equations for this system are then:

$$\dot{x}_1(t) = x_2(t),$$

$$\dot{x}_2(t) = \frac{-g \sin(x_1(t)) - Rx_2(t)^2}{(l + Rx_1(t))}, \quad \text{and}$$

$$y(t) = x_1(t).$$

- b. To linearize a state space model with two state variables $x_1(t)$ and $x_2(t)$ and one output variable $y(t)$, i.e.,:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} f_1(\mathbf{x}(t)) \\ f_2(\mathbf{x}(t)) \end{bmatrix},$$

$$y(t) = g(\mathbf{x}(t)),$$

around an operating point (x_{1Q}, x_{2Q}, y_Q) , we define three new variables, $\Delta x_1(t) = x_1(t) - x_{1Q}$, $\Delta x_2(t) = x_2(t) - x_{2Q}$, and $\Delta y(t) = y(t) - y_Q$, and use the following 1st-order Taylor series approximations:

$$\Delta \dot{\mathbf{x}}(t) = \begin{bmatrix} \Delta \dot{x}_1(t) \\ \Delta \dot{x}_2(t) \end{bmatrix} \approx \begin{bmatrix} \left. \frac{\partial f_1}{\partial x_1} \right|_{\substack{x=x_Q \\ u=u_Q}} & \left. \frac{\partial f_1}{\partial x_2} \right|_{\substack{x=x_Q \\ u=u_Q}} \\ \left. \frac{\partial f_2}{\partial x_1} \right|_{\substack{x=x_Q \\ u=u_Q}} & \left. \frac{\partial f_2}{\partial x_2} \right|_{\substack{x=x_Q \\ u=u_Q}} \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix},$$

$$\Delta y(t) \approx \begin{bmatrix} \left. \frac{\partial g}{\partial x_1} \right|_{\substack{x=x_Q \\ u=u_Q}} & \left. \frac{\partial g}{\partial x_2} \right|_{\substack{x=x_Q \\ u=u_Q}} \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix}.$$

For the state space equations derived in part a, these yield:

$$\Delta \dot{x}_1(t) = \Delta x_2(t),$$

$$\Delta \dot{x}_2(t) = \left[-\frac{g \cos(x_{1Q})}{(l + Rx_{1Q})} + \frac{Rg \sin(x_{1Q})}{(l + Rx_{1Q})^2} + \frac{(Rx_{2Q})^2}{(l + Rx_{1Q})^2} \right] \Delta x_1(t) - \frac{2Rx_{2Q}}{(l + Rx_{1Q})} \Delta x_2(t), \quad \text{and}$$

$$\Delta y(t) = \Delta x_1(t).$$

The first and last of these equations are independent of the operating point. Evaluating the second equation for the given operating point $\theta = 0$, $\dot{\theta} = 0 \Rightarrow x_{1Q} = x_{2Q} = y_Q = 0$, gives:

$$\Delta \dot{x}_2(t) = \left[-\frac{g \cos(0)}{(l + R \cdot 0)} + \frac{Rg \sin(0)}{(l + R \cdot 0)^2} + \frac{(R \cdot 0)^2}{(l + \cdot 0)^2} \right] \Delta x_1(t) - \frac{2R \cdot 0}{(l + R \cdot 0)} \Delta x_2(t)$$

$$= -\frac{g}{l} \Delta x_1(t)$$

Substituting our definitions of $\Delta x_1(t)$ and $\Delta x_2(t)$ into this equation produces:

$$\ddot{\theta} = -\frac{g}{l} \theta \Rightarrow \ddot{\theta} + (g/l)\theta = 0,$$

the equation for a simple pendulum.

[The equations for $\Delta \dot{x}_1(t)$ and $\Delta y(t)$ follow directly from the definitions and provide no further information about the behaviour of the system.]

2. In a nominal control loop, the complimentary sensitivity is given by:

$$T_o(s) = \frac{1}{(s+1)(s+10)}.$$

If the system has an input disturbance $d_i(t) = 2\sin(0.5t)$, what does this input disturbance contribute to the plant input $u(t)$ in the steady state? (25 pts)

From the one-d.o.f. closed-loop system equations, the contribution of an input disturbance to the plant output in the steady state is determined by:

$$U(s) = -T_o(s)D_i(s).$$

For a sinusoidal disturbance, this equation is most easily solved by phasor analysis, i.e.:

$$u(t) = |-T_o(j0.5)| \times 2 \sin(0.5t + \angle(-T_o(j0.5))).$$

Substituting for $s = j0.5$ in the equation for the nominal complimentary sensitivity gives:

$$T_o(j0.5) = \frac{1}{(j0.5+1)(j0.5+10)} \Rightarrow |-T_o(j0.5)| = 0.0893 \text{ and } \angle(-T_o(j0.5)) = 2.6280 \text{ rad,}$$

yielding:

$$u(t) = 0.1787 \sin(0.5t + 2.6280).$$

Alternatively, $U(s)$ can be calculated by taking the Laplace transform of $d_i(t)$:

$$U(s) = -T_o(s)D_i(s) = \frac{-1}{(s+1)(s+10)} \cdot \frac{1}{s^2 + 0.5^2}$$

$u(t)$ can then be obtained from the inverse Laplace transform of $U(s)$:

$$\begin{aligned} u(t) &= \mathcal{L}^{-1}[U(s)] = \mathcal{L}^{-1}\left[\frac{-1}{(s+1)(s+10)(s^2 + 0.5^2)}\right] \\ &= -\frac{4}{45}e^{-t} + \frac{4}{3609}e^{-10t} + \frac{176}{2005}\cos(0.5t) - \frac{312}{2005}\sin(0.5t) \end{aligned}$$

The exponential terms go to zero as $t \rightarrow \infty$, leaving the sinusoidal terms. The sum of these sinusoids can be found via the Fourier transform:

$$U(\omega) = \mathcal{F}\left[\frac{176}{2005}\cos(0.5t) - \frac{312}{2005}\sin(0.5t)\right] = 0.1787e^{-j2.6280} \times j\pi[\delta(\omega + 0.5) - \delta(\omega - 0.5)]$$

Making use of the ‘‘Delay property’’ of the Fourier transform, the inverse Fourier transform gives:

$$u(t) = 0.1787 \sin(0.5t + 2.6280).$$

3. Find the impulse response of the following linear transfer function:

$$H(s) = \frac{12}{(s+1)(s+3)(s+4)}. \quad (25 \text{ pts})$$

The impulse response $h(t)$ of the system is the inverse Laplace transform of the transfer function. Partial fraction decomposition of the given transfer function produces:

$$H(s) = \frac{2}{(s+1)} - \frac{6}{(s+3)} + \frac{4}{(s+4)}$$

$$\Rightarrow h(t) = \mathcal{L}^{-1}[H(s)] = \mathcal{L}^{-1}\left[\frac{2}{(s+1)} - \frac{6}{(s+3)} + \frac{4}{(s+4)}\right]$$

$$= 2e^{-t} - 6e^{-3t} + 4e^{-4t}$$

4. Find the range of values of K under which the controller:

$$C(s) = \frac{K(s+2)}{(s+10)}$$

stabilizes the unstable nominal plant model:

$$G_o(s) = \frac{1}{s(s-1)},$$

when placed together in a one-degree-of-freedom unity-feedback loop. (25 pts)

For this plant and controller, the characteristic polynomial $A_{cl}(s) = A_o(s)L(s) + B_o(s)P(s) = s(s-1)(s+10) + K(s+2) = s^3 + 9s^2 + (K-10)s + 2K$, and consequently Routh's array is:

s^3	1	$K-10$
s^2	9	$2K$
s^1	$\frac{7}{9}K - 10$	
s^0	$2K$	

For closed-loop stability, Routh's criterion states that there must be no changes of sign in the first column of the array. Consequently, $\frac{7}{9}K - 10 > 0$ and $2K > 0 \Rightarrow K > 90/7 (\approx 12.8571)$.

5. Consider a system having the following calibration and nominal models:

$$G(s) = F(s) \frac{1}{s-1} \quad \text{and} \quad G_o(s) = F(s) \frac{2}{s-2},$$

where $F(s)$ is a proper, stable, and minimum-phase transfer function. Prove the following:

- $G_\Delta(2) = -1$, i.e., the multiplicative modeling error equals -1 when $s = 2$.
- The error sensitivity, $S_\Delta(s) = \frac{1}{1 + T_o(s)G_\Delta(s)}$, is unstable, having a pole at $s = 2$, where $T_o(s)$ is the complementary sensitivity of an internally stable control loop.
- The achieved sensitivity $S(s) = S_\Delta(s)S_o(s)$ can be stable even though $S_\Delta(s)$ is unstable. (25 pts)

a. The MME for this system is:

$$G_\Delta(s) = \frac{G(s)}{G_o(s)} - 1 = \frac{s-2}{2(s-1)} - 1 = \frac{-s}{2(s-1)}.$$

Letting $s = 2$ yields $G_\Delta(2) = -1$.

b. Since the nominal loop is internally stable, then $C(s)$ cannot cancel the pole of $G_o(s)$ located at $s = 2$. Consequently, $S_o(2) = 0$ and $T_o(2) = 1 - S_o(2) = 1$. We have proven in part a that $G_\Delta(2) = -1$, so we can evaluate the error sensitivity at $s = 2$:

$$S_\Delta(2) = \frac{1}{1 + T_o(2)G_\Delta(2)} = \frac{1}{1 + (1) \cdot (-1)} = \infty.$$

Thus, the error sensitivity is unstable having a pole at $s = 2$.

An alternative solution is:

$$\begin{aligned} S_\Delta(s) &= \frac{S(s)}{S_o(s)} = \frac{1 + G_o(s)C(s)}{1 + G(s)C(s)} = \frac{1 + C(s)F(s) \frac{2}{s-2}}{1 + C(s)F(s) \frac{1}{s-1}} = \frac{s-2 + 2C(s)F(s)}{s-2} \cdot \frac{s-1}{s-1 + C(s)F(s)} \\ &= \frac{(s-1)(s-2 + 2C(s)F(s))}{(s-2)(s-1 + C(s)F(s))}, \end{aligned}$$

which has an unstable pole at $s = 2$.

c. We also notice that, since $S_o(s)$ has a zero at $s = 2$, the achieved sensitivity $S(s) = S_\Delta(s)S_o(s)$ has no pole at $s = 2$, since the pole of $S_\Delta(s)$ at $s = 2$ is cancelled by the zero of $S_o(s)$ at the same location.