Tables and Other Information

State-space model equations:

for continuous-time systems,

$$\frac{dx}{dt} = f(x(t), u(t), t) \tag{3.6.1}$$
$$y(t) = g(x(t), u(t), t) \tag{3.6.2}$$

for discrete-time systems,

$$x[k+1] = f_d(x[k], u[k], k) \tag{3.6.3}$$
$$y[k] = g_d(x[k], u[k], k) \tag{3.6.4}$$

In the linear, time-invariant case, equations (3.6.1) and (3.6.2) become

$$\frac{dx(t)}{dt} = \mathbf{A}x(t) + \mathbf{B}u(t) \tag{3.6.5}$$
$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t) \tag{3.6.6}$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and $\mathbf{D}$ are matrices of appropriate dimensions.

Linearization of state-space models:

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t) + \mathbf{E} \tag{3.10.8}$$
$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t) + \mathbf{F} \tag{3.10.9}$$

where

$$\mathbf{A} = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}}; \qquad \mathbf{B} = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}} \tag{3.10.10}$$
$$\mathbf{C} = \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}}; \qquad \mathbf{D} = \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}} \tag{3.10.11}$$
$$\mathbf{E} = f(x_Q, u_Q) - \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}} x_Q - \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}} u_Q \tag{3.10.12}$$
$$\mathbf{F} = g(x_Q, u_Q) - \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}} x_Q - \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}} u_Q \tag{3.10.13}$$

Laplace-transform definition:

$$\mathcal{L}[y(t)] = Y(s) = \int_{0^-}^{\infty} e^{-st} y(t) dt \qquad (4.3.1)$$

$$\mathcal{L}^{-1}[y(s)] = y(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} e^{st} Y(s) ds \qquad (4.3.2)$$

Laplace-transform table:

$f(t) \quad (t \geq 0)$	$\mathcal{L}[f(t)]$	Region of Convergence
1	$\frac{1}{s}$	$\sigma > 0$
$\delta_D(t)$	1	$\lvert\sigma\rvert < \infty$
t	$\frac{1}{s^2}$	$\sigma > 0$
$t^n \quad n \in \mathbb{Z}^+$	$\frac{n!}{s^{n+1}}$	$\sigma > 0$
$e^{\alpha t} \quad \alpha \in \mathbb{C}$	$\frac{1}{s - \alpha}$	$\sigma > \Re\{\alpha\}$
$te^{\alpha t} \quad \alpha \in \mathbb{C}$	$\frac{1}{(s - \alpha)^2}$	$\sigma > \Re\{\alpha\}$
$\cos(\omega_o t)$	$\frac{s}{s^2 + \omega_o^2}$	$\sigma > 0$
$\sin(\omega_o t)$	$\frac{\omega_o}{s^2 + \omega_o^2}$	$\sigma > 0$
$e^{\alpha t} \sin(\omega_o t + \beta)$	$\frac{(\sin\beta)s + \omega_o \cos\beta - \alpha \sin\beta}{(s - \alpha)^2 + \omega_o^2}$	$\sigma > \Re\{\alpha\}$
$t \sin(\omega_o t)$	$\frac{2\omega_o s}{(s^2 + \omega_o^2)^2}$	$\sigma > 0$
$t \cos(\omega_o t)$	$\frac{s^2 - \omega_o^2}{(s^2 + \omega_o^2)^2}$	$\sigma > 0$
$\mu(t) - \mu(t - \tau)$	$\frac{1 - e^{-s\tau}}{s}$	$\lvert\sigma\rvert < \infty$

Table 4.1. Laplace-transform table

Laplace-transform properties:

$f(t)$	$\mathcal{L}[f(t)]$	Names
$\sum_{i=1}^{l} a_i f_i(t)$	$\sum_{i=1}^{l} a_i F_i(s)$	Linear combination
$\frac{dy(t)}{dt}$	$sY(s) - y(0^-)$	Derivative Law
$\frac{d^k y(t)}{dt^k}$	$s^k Y(s) - \sum_{i=1}^{k} s^{k-i} \left.\frac{d^{i-1}y(t)}{dt^{i-1}}\right\vert_{t=0^-}$	High-order derivative
$\int_{0^-}^{t} y(\tau)d\tau$	$\frac{1}{s}Y(s)$	Integral Law
$y(t-\tau)\mu(t-\tau)$	$e^{-s\tau}Y(s)$	Delay
$ty(t)$	$-\frac{dY(s)}{ds}$	
$t^k y(t)$	$(-1)^k \frac{d^k Y(s)}{ds^k}$	
$\int_{0^-}^{t} f_1(\tau) f_2(t-\tau)d\tau$	$F_1(s)F_2(s)$	Convolution
$\lim_{t\to\infty} y(t)$	$\lim_{s\to 0} sY(s)$	Final-Value Theorem
$\lim_{t\to 0^+} y(t)$	$\lim_{s\to\infty} sY(s)$	Initial Value Theorem
$f_1(t)f_2(t)$	$\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F_1(\zeta)F_2(s-\zeta)d\zeta$	Time-domain product
$e^{at}f_1(t)$	$F_1(s-a)$	Frequency Shift

Table 4.2. Laplace-transform properties–note that $F_i(s) = \mathcal{L}[f_i(t)]$, $Y(s) = \mathcal{L}[y(t)]$, $k \in \{1,2,3,\ldots\}$, and $f_1(t) = f_2(t) = 0 \quad \forall t < 0$.

Fourier-transform definition:

$$\mathcal{F}[f(t)] = F(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} f(t)dt \qquad (4.10.1)$$

$$\mathcal{F}^{-1}[F(j\omega)] = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} F(j\omega)d\omega \qquad (4.10.2)$$

Fourier-transform table:

$f(t) \qquad \forall t \in \mathbb{R}$	$\mathcal{F}[f(t)]$
1	$2\pi\delta(\omega)$
$\delta_D(t)$	1
$\mu(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\mu(t) - \mu(t - t_o)$	$\frac{1 - e^{-j\omega t_o}}{j\omega}$
$e^{\alpha t}\mu(t) \qquad \Re\{\alpha\} < 0$	$\frac{1}{j\omega - \alpha}$
$te^{\alpha t}\mu(t) \qquad \Re\{\alpha\} < 0$	$\frac{1}{(j\omega - \alpha)^2}$
$e^{-\alpha\|t\|} \qquad \alpha \in \mathbb{R}^+$	$\frac{2\alpha}{\omega^2 + \alpha^2}$
$\cos(\omega_o t)$	$\pi\left(\delta(\omega - \omega_o) + \delta(\omega - \omega_o)\right)$
$\sin(\omega_o t)$	$j\pi\left(\delta(\omega + \omega_o) - \delta(\omega - \omega_o)\right)$
$\cos(\omega_o t)\mu(t)$	$\pi\left(\delta(\omega - \omega_o) + \delta(\omega - \omega_o)\right) + \frac{j\omega}{-\omega^2 + \omega_o^2}$
$\sin(\omega_o t)\mu(t)$	$j\pi\left(\delta(\omega + \omega_o) - \delta(\omega - \omega_o)\right) + \frac{\omega_o}{-\omega^2 + \omega_o^2}$
$e^{-\alpha t}\cos(\omega_o t)\mu(t) \qquad \alpha \in \mathbb{R}^+$	$\frac{j\omega + \alpha}{(j\omega + \alpha)^2 + \omega_o^2}$
$e^{-\alpha t}\sin(\omega_o t)\mu(t) \qquad \alpha \in \mathbb{R}^+$	$\frac{\omega_o}{(j\omega + \alpha)^2 + \omega_o^2}$

Table 4.3. Fourier transform table

Fourier-transform properties:

$f(t)$	$\mathcal{F}[f(t)]$	Description
$\sum_{i=1}^{l} a_i f_i(t)$	$\sum_{i=1}^{l} a_i F_i(j\omega)$	Linearity
$\frac{dy(t)}{dt}$	$j\omega Y(j\omega)$	Derivative law
$\frac{d^k y(t)}{dt^k}$	$(j\omega)^k Y(j\omega)$	High-order derivative
$\int_{-\infty}^{t} y(\tau)d\tau$	$\frac{1}{j\omega}Y(j\omega) + \pi Y(0)\delta(\omega)$	Integral law
$y(t-\tau)$	$e^{-j\omega\tau}Y(j\omega)$	Delay
$y(at)$	$\frac{1}{\lvert a\rvert}Y\left(j\frac{\omega}{a}\right)$	Time scaling
$y(-t)$	$Y(-j\omega)$	Time reversal
$\int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau$	$F_1(j\omega)F_2(j\omega)$	Convolution
$y(t)\cos(\omega_o t)$	$\frac{1}{2}\{Y(j\omega - j\omega_o) + Y(j\omega + j\omega_o)\}$	Modulation (cosine)
$y(t)\sin(\omega_o t)$	$\frac{1}{j2}\{Y(j\omega - j\omega_o) - Y(j\omega + j\omega_o)\}$	Modulation (sine)
$F(t)$	$2\pi f(-j\omega)$	Symmetry
$f_1(t)f_2(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(j\zeta)F_2(j\omega - j\zeta)d\zeta$	Time-domain product
$e^{at}f_1(t)$	$F_1(j\omega - a)$	Frequency shift

Table 4.4. Fourier transform properties; note that $F_i(j\omega) = \mathcal{F}[f_i(t)]$ and $Y(j\omega) = \mathcal{F}[y(t)]$

Nominal sensitivity equations:

$$T_o(s) \triangleq \frac{G_o(s)C(s)}{1+G_o(s)C(s)} = \frac{B_o(s)P(s)}{A_o(s)L(s)+B_o(s)P(s)} \tag{5.3.1}$$

$$S_o(s) \triangleq \frac{1}{1+G_o(s)C(s)} = \frac{A_o(s)L(s)}{A_o(s)L(s)+B_o(s)P(s)} \tag{5.3.2}$$

$$S_{io}(s) \triangleq \frac{G_o(s)}{1+G_o(s)C(s)} = \frac{B_o(s)L(s)}{A_o(s)L(s)+B_o(s)P(s)} \tag{5.3.3}$$

$$S_{uo}(s) \triangleq \frac{C(s)}{1+G_o(s)C(s)} = \frac{A_o(s)P(s)}{A_o(s)L(s)+B_o(s)P(s)} \tag{5.3.4}$$

$T_o(s)$:	Nominal complementary sensitivity
$S_o(s)$:	Nominal sensitivity
$S_{io}(s)$:	Nominal input-disturbance sensitivity
$S_{uo}(s)$:	Nominal control sensitivity

The polynomial $A_{cl} \triangleq A_o(s)L(s) + B_o(s)P(s)$ is called the nominal closed-loop characteristic polynomial.

$$S_o(s) + T_o(s) = 1 \tag{5.3.5}$$

$$S_{io}(s) = S_o(s)G_o(s) = \frac{T_o(s)}{C(s)} \tag{5.3.6}$$

$$S_{uo}(s) = S_o(s)C(s) = \frac{T_o(s)}{G_o(s)} \tag{5.3.7}$$

One-d.o.f. closed-loop system equations:

$$\begin{bmatrix} Y(s) \\ U(s) \end{bmatrix} = \frac{\begin{bmatrix} G_o(s)C(s) & G_o(s) & 1 & -G_o(s)C(s) \\ C(s) & -G_o(s)C(s) & -C(s) & -C(s) \end{bmatrix}}{1+G_o(s)C(s)} \begin{bmatrix} R(s) \\ D_i(s) \\ D_o(s) \\ D_m(s) \end{bmatrix}$$

Two-d.o.f. closed-loop system equations:

$$\begin{bmatrix} Y_o(s) \\ U_o(s) \end{bmatrix} = \frac{\begin{bmatrix} G_o(s)C(s) & G_o(s) & 1 & -G_o(s)C(s) \\ C(s) & -G_o(s)C(s) & -C(s) & -C(s) \end{bmatrix}}{1+G_o(s)C(s)} \begin{bmatrix} H(s)R(s) \\ D_i(s) \\ D_o(s) \\ D_m(s) \end{bmatrix} \tag{5.3.12}$$

Routh's array:

$\mathbf{s^n}$	$\gamma_{0,1}$	$\gamma_{0,2}$	$\gamma_{0,3}$	$\gamma_{0,4}$	$\cdots$
$\mathbf{s^{n-1}}$	$\gamma_{1,1}$	$\gamma_{1,2}$	$\gamma_{1,3}$	$\gamma_{1,4}$	$\cdots$
$\mathbf{s^{n-2}}$	$\gamma_{2,1}$	$\gamma_{2,2}$	$\gamma_{2,3}$	$\gamma_{2,4}$	$\cdots$
$\mathbf{s^{n-3}}$	$\gamma_{3,1}$	$\gamma_{3,2}$	$\gamma_{3,3}$	$\gamma_{3,4}$	$\cdots$
$\mathbf{s^{n-4}}$	$\gamma_{4,1}$	$\gamma_{4,2}$	$\gamma_{4,,3}$	$\gamma_{4,4}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\mathbf{s^2}$	$\gamma_{n-2,1}$	$\gamma_{n-2,2}$			
$\mathbf{s^1}$	$\gamma_{n-1,1}$				
$\mathbf{s^0}$	$\gamma_{n,1}$				

Table 5.1. Routh's array

where

$$\gamma_{0,i} = a_{n+2-2i}; \quad i = 1, 2, \ldots, m_0 \qquad \text{and} \quad \gamma_{1,i} = a_{n+1-2i}; \quad i = 1, 2, \ldots, m_1 \tag{5.5.9}$$

with $m_0 = (n+2)/2$ and $m_1 = m_0 - 1$ for n even and $m_1 = m_0$ for n odd. Note that the elements $\gamma_{0,i}$ and $\gamma_{1,i}$ are the coefficients of the polynomials arranged in alternated form.

Furthermore,

$$\gamma_{k,j} = \frac{\gamma_{k-1,1}\,\gamma_{k-2,j+1} - \gamma_{k-2,1}\,\gamma_{k-1,j+1}}{\gamma_{k-1,1}}; \qquad k = 2, \ldots, n \qquad j = 1, 2, \ldots, m_j \tag{5.5.10}$$

where $m_j = \max\{m_{j-1}, m_{j-2}\} - 1$ and where we must assign a zero value to the coefficient $\gamma_{k-1,j+1}$ when it is not defined in the Routh's array given in Table 5.1.

Note that the definitions of coefficients in (5.5.10) can be expressed by using determinants:

$$\gamma_{k,j} = -\frac{1}{\gamma_{k-1,1}} \begin{vmatrix} \gamma_{k-2,1} & \gamma_{k-2,j+1} \\ \gamma_{k-1,1} & \gamma_{k-1,j+1} \end{vmatrix} \tag{5.5.11}$$

Robust stability theorm:

Theorem 5.3 (Robust stability theorem). *Consider a plant with nominal transfer function $G_o(s)$ and true transfer function given by $G(s)$. Assume that $C(s)$ is the transfer function of a controller that achieves nominal internal stability. Also assume that $G(s)C(s)$ and $G_o(s)C(s)$ have the same number of unstable poles. Then, a sufficient condition for stability of the true feedback loop obtained by applying the controller to the true plant is that*

$$|T_o(j\omega)||G_\Delta(j\omega)| = \left|\frac{G_o(j\omega)C(j\omega)}{1+G_o(j\omega)C(j\omega)}\right||G_\Delta(j\omega)| < 1 \qquad \forall\omega \tag{5.9.6}$$

where $G_\Delta(j\omega)$ is the frequency response of the multiplicative modeling error (MME).

Achieved sensitivity equations:

$$T(s) \triangleq \frac{G(s)C(s)}{1+G(s)C(s)} = \frac{B(s)P(s)}{A(s)L(s)+B(s)P(s)} \tag{5.9.1}$$

$$S(s) \triangleq \frac{1}{1+G(s)C(s)} = \frac{A(s)L(s)}{A(s)L(s)+B(s)P(s)} \tag{5.9.2}$$

$$S_i(s) \triangleq \frac{G(s)}{1+G(s)C(s)} = \frac{B(s)L(s)}{A(s)L(s)+B(s)P(s)} \tag{5.9.3}$$

$$S_u(s) \triangleq \frac{C(s)}{1+G(s)C(s)} = \frac{A(s)P(s)}{A(s)L(s)+B(s)P(s)} \tag{5.9.4}$$

$$S(s) = S_o(s)S_\Delta(s) \tag{5.9.15}$$

$$T(s) = T_o(s)(1+G_\Delta(s))S_\Delta(s) \tag{5.9.16}$$

$$S_i(s) = S_{io}(s)(1+G_\Delta(s))S_\Delta(s) \tag{5.9.17}$$

$$S_u(s) = S_{uo}(s)S_\Delta(s) \tag{5.9.18}$$

$$S_\Delta(s) = \frac{1}{1+T_o(s)G_\Delta(s)} \tag{5.9.19}$$

$S_\Delta(s)$ *is called the* error sensitivity.

PID standard form equations:

$$C_P(s) = K_p \tag{6.2.1}$$

$$C_{PI}(s) = K_p \left(1 + \frac{1}{T_r s}\right) \tag{6.2.2}$$

$$C_{PD}(s) = K_p \left(1 + \frac{T_d s}{\tau_D s + 1}\right) \tag{6.2.3}$$

$$C_{PID}(s) = K_p \left(1 + \frac{1}{T_r s} + \frac{T_d s}{\tau_D s + 1}\right) \tag{6.2.4}$$

Ziegler-Nichols oscillation-method parameters:

	$\mathbf{K_p}$	$\mathbf{T_r}$	$\mathbf{T_d}$
P	$0.50K_c$		
PI	$0.45K_c$	$\frac{P_c}{1.2}$	
PID	$0.60K_c$	$0.5P_c$	$\frac{P_c}{8}$

Table 6.1. Ziegler-Nichols tuning, using the oscillation method

Ziegler-Nichols reaction curve parameters:

	$\mathbf{K_p}$	$\mathbf{T_r}$	$\mathbf{T_d}$
P	$\frac{\nu_o}{K_o \tau_o}$		
PI	$\frac{0.9\nu_o}{K_o \tau_o}$	$3\tau_o$	
PID	$\frac{1.2\nu_o}{K_o \tau_o}$	$2\tau_o$	$0.5\tau_o$

Table 6.2. Ziegler-Nichols tuning by using the reaction curve

Cohen-Coon reaction curve parameters:

	$\mathbf{K_p}$	$\mathbf{T_r}$	$\mathbf{T_d}$
P	$\frac{\nu_o}{K_o\tau_o}\left[1+\frac{\tau_o}{3\nu_o}\right]$		
PI	$\frac{\nu_o}{K_o\tau_o}\left[0.9+\frac{\tau_o}{12\nu_o}\right]$	$\frac{\tau_o[30\nu_o+3\tau_o]}{9\nu_o+20\tau_o}$	
PID	$\frac{\nu_o}{K_o\tau_o}\left[\frac{4}{3}+\frac{\tau_o}{4\nu_o}\right]$	$\frac{\tau_o[32\nu_o+6\tau_o]}{13\nu_o+8\tau_o}$	$\frac{4\tau_o\nu_o}{11\nu_o+2\tau_o}$

Table 6.3. Cohen-Coon tuning by using the reaction curve

Lead-lag compensator equation:

$$C(s) = \frac{\tau_1 s + 1}{\tau_2 s + 1} \qquad (6.6.1)$$

Pole placement equations and Lemma:

$$C(s) = \frac{P(s)}{L(s)} \qquad G_o(s) = \frac{B_o(s)}{A_o(s)} \qquad (7.2.2)$$

$$P(s) = p_{n_p}s^{n_p} + p_{n_p-1}s^{n_p-1} + \ldots + p_0 \qquad (7.2.3)$$

$$L(s) = l_{n_l}s^{n_l} + l_{n_l-1}s^{n_l-1} + \ldots + l_0 \qquad (7.2.4)$$

$$B_o(s) = b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_0 \qquad (7.2.5)$$

$$A_o(s) = a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0 \qquad (7.2.6)$$

$$A_{cl}(s) = a^c_{n_c}s^{n_c} + a^c_{n_c-1}s^{n_c-1} + \ldots + a^c_0 \qquad (7.2.7)$$

Lemma 7.1 (SISO pole placement–polynomial approach). *Consider a one-d.o.f. feedback loop with controller and plant nominal model given by (7.2.2) to (7.2.6). Assume that $B_o(s)$ and $A_o(s)$ are relatively prime (coprime)–i.e., they have no common factors. Let $A_{cl}(s)$ be an arbitrary polynomial of degree $n_c = 2n - 1$. Then there exist polynomials $P(s)$ and $L(s)$, with degrees $n_p = n_l = n - 1$, such that*

$$A_o(s)L(s) + B_o(s)P(s) = A_{cl}(s) \qquad (7.2.19)$$

Continued on page 14

Time-domain integral constraints:

Lemma 8.1. *We assume that the plant is controlled in a one-degree-of-freedom configuration and that the open-loop plant and controller satisfy:*

$$A_o(s)L(s) = s^i(A_o(s)L(s))' \qquad i \geq 1 \tag{8.6.8}$$

$$\lim_{s \to 0}(A_o(s)L(s))' = c_0 \neq 0 \tag{8.6.9}$$

$$\lim_{s \to 0}(B_o(s)P(s)) = c_1 \neq 0 \tag{8.6.10}$$

that is, that the plant-controller combination has i poles at the origin. Then, for a step-output disturbance or step set-point, the control error, $e(t)$, *satisfies*

$$\lim_{t \to \infty} e(t) = 0 \qquad \forall i \geq 1 \tag{8.6.11}$$

$$\int_0^\infty e(t)dt = 0 \qquad \forall i \geq 2 \tag{8.6.12}$$

Also, for a negative unit ramp output disturbance or a positive unit ramp reference, the control error, $e(t)$, *satisfies*

$$\lim_{t \to \infty} e(t) = \frac{c_0}{c_1} \qquad \textit{for} \quad i = 1 \tag{8.6.13}$$

$$\lim_{t \to \infty} e(t) = 0 \qquad \forall i \geq 2 \tag{8.6.14}$$

$$\int_0^\infty e(t)dt = 0 \qquad \forall i \geq 3 \tag{8.6.15}$$

Continued on page 14

Time-domain integral constraints (cont.):

Lemma 8.2. *Assume that the controller satisfies:*

$$L(s) = s^i (L(s))' \qquad i \geq 1 \tag{8.6.17}$$

$$\lim_{s \to 0} (L(s))' = l_i \neq 0 \tag{8.6.18}$$

$$\lim_{s \to 0} (P(s)) = p_0 \neq 0 \tag{8.6.19}$$

The controller alone has i poles at the origin. Then, for a step input disturbance, the control error, $e(t)$, satisfies

$$\lim_{t \to \infty} e(t) = 0 \qquad \forall i \geq 1 \tag{8.6.20}$$

$$\int_0^\infty e(t)dt = 0 \qquad \forall i \geq 2 \tag{8.6.21}$$

Also, for a negative unit ramp input disturbance, the control error, $e(t)$, satisfies

$$\lim_{t \to \infty} e(t) = \frac{l_i}{p_0} \qquad \text{for} \qquad i = 1 \tag{8.6.22}$$

$$\lim_{t \to \infty} e(t) = 0 \qquad \forall i \geq 2 \tag{8.6.23}$$

$$\int_0^\infty e(t)dt = 0 \qquad \forall i \geq 3 \tag{8.6.24}$$

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Time-domain integral constraints (cont.):

Lemma 8.3. *Consider a feedback control loop having stable closed-loop poles located to the left of $-\alpha$ for some $\alpha > 0$. Also assume that the controller has at least one pole at the origin. Then, for an uncancelled plant zero z_0 or an uncancelled plant pole η_0 to the right of the closed-loop poles–i.e., satisfying $\Re\{z_0\} > -\alpha$ or $\Re\{\eta_0\} > -\alpha$, respectively–we have the following:*

(i) For a positive unit reference step or a negative unit-step output disturbance, we have

$$\int_0^\infty e(t)e^{-z_0 t}dt = \frac{1}{z_0} \tag{8.6.26}$$

$$\int_0^\infty e(t)e^{-\eta_0 t}dt = 0 \tag{8.6.27}$$

(ii) For a positive unit step reference and for z_0 in the right-half plane, we have

$$\int_0^\infty y(t)e^{-z_0 t}dt = 0 \tag{8.6.28}$$

(iii) For a negative unit step input disturbance, we have

$$\int^\infty e(t)e^{-z_0 t}dt = 0 \tag{8.6.29}$$

$$\int_0^\infty e(t)e^{-\eta_0 t}dt = \frac{L(\eta_0)}{\eta_0 P(\eta_0)} \tag{8.6.30}$$

Corollary 8.1. *Consider a closed-loop system, as in Lemma 8.3 on page 213. Then, for a unit step reference input,*

(a) if the plant $G(s)$ has a pair of zeros at $\pm j\omega_0$, then

$$\int_0^\infty e(t)\cos\omega_0 t dt = 0 \tag{8.6.39}$$

$$\int_0^\infty e(t)\sin\omega_0 t dt = \frac{1}{\omega_0} \tag{8.6.40}$$

(b) if the plant $G(s)$ has a pair of poles at $\pm j\omega_0$, then

$$\int_0^\infty e(t)\cos\omega_0 t dt = 0 \tag{8.6.41}$$

$$\int_0^\infty e(t)\sin\omega_0 t dt = 0 \tag{8.6.42}$$

where $e(t)$ is the control error, i.e.,

$$e(t) = 1 - y(t) \tag{8.6.43}$$

Z-transform definition:

$$\mathcal{Z}\left[y[k]\right] = Y(z) = \sum_{k=0}^{\infty} z^{-k} y[k] \tag{12.6.1}$$

$$\mathcal{Z}^{-1}\left[Y(z)\right] = y[k] = \frac{1}{2\pi j} \oint z^{k-1} Y(z) dz \tag{12.6.2}$$

Z-transform table:

$f[k]$	$\mathcal{Z}\left[f[k]\right]$	Region of convergence
1	$\dfrac{z}{z-1}$	$\lvert z\rvert > 1$
$\delta_K[k]$	1	$\lvert z\rvert > 0$
k	$\dfrac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
k^2	$\dfrac{z(z+1)}{(z-1)^3}$	$\lvert z\rvert > 1$
a^k	$\dfrac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
ka^k	$\dfrac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
$\cos k\theta$	$\dfrac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$	$\lvert z\rvert > 1$
$\sin k\theta$	$\dfrac{z\sin\theta}{z^2-2z\cos\theta+1}$	$\lvert z\rvert > 1$
$a^k \cos k\theta$	$\dfrac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}$	$\lvert z\rvert > a$
$a^k \sin k\theta$	$\dfrac{az\sin\theta)}{z^2-2az\cos\theta+a^2}$	$\lvert z\rvert > a$
$k\cos k\theta$	$\dfrac{z(z^2\cos\theta-2z+\cos\theta)}{z^2-2z\cos\theta+1}$	$\lvert z\rvert > 1$
$\mu[k]-\mu[k-k_o],\quad k_o \in \mathbb{N}$	$\dfrac{1+z+z^2+\ldots+z^{k_o-1}}{z^{k_o-1}}$	$\lvert z\rvert > 0$

Table 12.1. Z-transform table

Z-transform properties:

$f[k]$	$\mathcal{Z}[f[k]]$	Names
$\sum_{i=1}^{l} a_i f_i[k]$	$\sum_{i=1}^{l} a_i F_i(z)$	Partial fractions
$f[k+1]$	$zF(z) - zf(0)$	Forward shift
$\sum_{l=0}^{k} f[l]$	$\frac{z}{z-1}F(z)$	Summation
$f[k-1]$	$z^{-1}F(z) + f(-1)$	Backward shift
$y[k-l]\mu[k-l]$	$z^{-l}Y(z)$	Unit step
$kf[k]$	$-z\frac{dF(z)}{dz}$	
$\frac{1}{k}f[k]$	$\int_z^\infty \frac{F(\zeta)}{\zeta}d\zeta$	
$\lim_{k\to\infty} y[k]$	$\lim_{z\to 1}(z-1)Y(z)$	Final-value theorem
$\lim_{k\to 0} y[k]$	$\lim_{z\to\infty} Y(z)$	Initial value theorem
$\sum_{l=0}^{k} f_1[l]f_2[k-l]$	$F_1(z)F_2(z)$	Convolution
$f_1[k]f_2[k]$	$\frac{1}{2\pi j}\oint F_1(\zeta)F_2\left(\frac{z}{\zeta}\right)\frac{d\zeta}{\zeta}$	Complex convolution
$(\lambda)^k f_1[k]$	$F_1\left(\frac{z}{\lambda}\right)$	Frequency scaling

Table 12.2. Z-transform properties. Note that $F_i(z) = \mathcal{Z}[f_i[k]]$, that $\mu[k]$ denotes, as usual, a unit step, that $y[\infty]$ must be well defined, and that the convolution property holds (provided that $f_1[k] = f_2[k] = 0$ for all $k < 0$).

Discrete delta-transform definition:

$$\mathcal{D}\left[y(k\Delta)\right] \triangleq Y_\delta(\gamma) = \sum_{k=0}^{\infty}(1+\gamma\Delta)^{-k}y(k\Delta)\Delta \qquad (12.9.5)$$

$$\mathcal{D}^{-1}\left[Y_\delta(\gamma)\right] = y(k\Delta) = \frac{1}{2\pi j}\oint(1+\gamma\Delta)^{k-1}Y_\delta(\gamma)d\gamma \qquad (12.9.6)$$

Discrete delta-transform table:

$f[k] \quad (k \geq 0)$	$\mathcal{D}\left[f[k]\right]$	Region of Convergence
1	$\frac{1+\Delta\gamma}{\gamma}$	$\left\lvert\gamma+\frac{1}{\Delta}\right\rvert > \frac{1}{\Delta}$
$\frac{1}{\Delta}\delta_K[k]$	1	$\lvert\gamma\rvert < \infty$
$\mu[k]-\mu[k-1]$	$\frac{1}{\Delta}$	$\lvert\gamma\rvert < \infty$
k	$\frac{1+\Delta\gamma}{\Delta\gamma^2}$	$\left\lvert\gamma+\frac{1}{\Delta}\right\rvert > \frac{1}{\Delta}$
k^2	$\frac{(1+\Delta\gamma)(2+\Delta\gamma)}{\Delta^2\gamma^3}$	$\left\lvert\gamma+\frac{1}{\Delta}\right\rvert > \frac{1}{\Delta}$
$e^{\alpha\Delta k} \quad \alpha\in\mathbb{C}$	$\frac{1+\Delta\gamma}{\gamma-\frac{e^{\alpha\Delta}-1}{\Delta}}$	$\left\lvert\gamma+\frac{1}{\Delta}\right\rvert > \frac{e^{\alpha\Delta}}{\Delta}$
$ke^{\alpha\Delta k} \quad \alpha\in\mathbb{C}$	$\frac{(1+\Delta\gamma)e^{\alpha\Delta}}{\Delta\left(\gamma-\frac{e^{\alpha\Delta}-1}{\Delta}\right)^2}$	$\left\lvert\gamma+\frac{1}{\Delta}\right\rvert > \frac{e^{\alpha\Delta}}{\Delta}$
$\sin(\omega_o\Delta k)$	$\frac{(1+\Delta\gamma)\omega_o\mathrm{sinc}(\omega_o\Delta)}{\gamma^2+\Delta\phi(\omega_o,\Delta)\gamma+\phi(\omega_o,\Delta)}$ where $\mathrm{sinc}(\omega_o\Delta)=\frac{\sin(\omega_o\Delta)}{\omega_o\Delta}$ and $\phi(\omega_o,\Delta)=\frac{2(1-\cos(\omega_o\Delta))}{\Delta^2}$	$\left\lvert\gamma+\frac{1}{\Delta}\right\rvert > \frac{1}{\Delta}$
$\cos(\omega_o\Delta k)$	$\frac{(1+\Delta\gamma)(\gamma+0.5\Delta\phi(\omega_o,\Delta))}{\gamma^2+\Delta\phi(\omega_o,\Delta)\gamma+\phi(\omega_o,\Delta)}$	$\left\lvert\gamma+\frac{1}{\Delta}\right\rvert > \frac{1}{\Delta}$

Table 12.3. Delta-Transform table

Discrete delta-transform properties:

$f[k]$	$\mathcal{D}[f[k]]$	Names
$\sum_{i=1}^{l} a_i f_i[k]$	$\sum_{i=1}^{l} a_i F_i(\gamma)$	Partial fractions
$f_1[k+1]$	$(\Delta\gamma + 1)(F_1(\gamma) - f_1[0])$	Forward shift
$\frac{f_1[k+1] - f_1[k]}{\Delta}$	$\gamma F_1(\gamma) - (1 + \gamma\Delta) f_1[0]$	Scaled difference
$\sum_{l=0}^{k-1} f[l]\Delta$	$\frac{1}{\gamma} F(\gamma)$	Reimann sum
$f[k-1]$	$(1+\gamma\Delta)^{-1} F(\gamma) + f[-1]$	Backward shift
$f[k-l]\mu[k-l]$	$(1+\gamma\Delta)^{-l} F(\gamma)$	
$kf[k]$	$-\frac{1+\gamma\Delta}{\Delta} \frac{dF(\gamma)}{d\gamma}$	
$\frac{1}{k} f[k]$	$\int_{\gamma}^{\infty} \frac{F(\zeta)}{1+\zeta\Delta} d\zeta$	
$\lim_{k\to\infty} f[k]$	$\lim_{\gamma\to 0} \gamma F(\gamma)$	Final-value theorem
$\lim_{k\to 0} f[k]$	$\lim_{\gamma\to\infty} \frac{\gamma F(\gamma)}{1+\gamma\Delta}$	Initial value theorem
$\sum_{l=0}^{k-1} f_1[l] f_2[k-l]\Delta$	$F_1(\gamma) F_2(\gamma)$	Convolution
$f_1[k] f_2[k]$	$\frac{1}{2\pi j} \oint F_1(\zeta) F_2\left(\frac{\gamma - \zeta}{1+\zeta\Delta}\right) \frac{d\zeta}{1+\zeta\Delta}$	Complex convolution
$(1+a\Delta)^k f_1[k]$	$F_1\left(\frac{\gamma - a}{1+a\Delta}\right)$	

Table 12.4. Delta-Transform properties. Note that $F_i(\gamma) = \mathcal{D}[f_i[k]]$, that $\mu[k]$ denotes, as usual, a unit step, that $f[\infty]$ must be well defined, and that the convolution property holds (provided that $f_1[k] = f_2[k] = 0$ for all $k < 0$).

Discrete-time equivalent model with ZOH:

$$
\boxed{\begin{aligned}
H_{oq}(z) &= \mathcal{Z}\left[\text{the sampled impulse response of } G_{h0}(s)G_o(s)\right] && (12.13.3)\\
&= (1 - z^{-1})\mathcal{Z}\left[\text{the sampled step response of } G_o(s)\right] && (12.13.4)
\end{aligned}}
$$

"Approximate continuous" digital controller—direct:

$$\overline{C}_1(\gamma) = C(s)\big|_{s=\gamma} \qquad (13.5.1)$$

"Approximate continuous" digital controller—step invariant:

$$\overline{C}_2(\gamma) = \mathcal{D}\left[\text{sampled impulse response of } \{C(s)G_{h0}(s)\}\right] \qquad (13.5.2)$$

"Approximate continuous" digital controller—bilinear transformation:

$$\overline{C}_3(\gamma) = C(s)\big|_{s\,=\,\frac{\alpha\gamma}{\frac{\Delta}{2}\gamma+1}} \qquad (13.5.4)$$

"Minimal prototype" digital controller—strictly stable and minimum-phase plant:

$$C_q(z) = [G_{oq}(z)]^{-1}\frac{1}{z^{n-m}-1}; \qquad \text{and} \qquad T_o(z) = \frac{1}{z^{n-m}} \qquad (13.6.10)$$

"Minimal prototype" digital controller—pole at *z* = 1:

The plant is assumed to be of minimum phase and stable, except for a pole at $z = 1$, i.e., $A_{oq}(z) = (z-1)\overline{A}_{oq}(z)$.

$$
\begin{aligned}
C_q(z) &= [G_{oq}(z)]^{-1}\frac{1}{z^{n-m}-1} = \frac{\overline{A}_{oq}(z)}{B_{oq}(z)}\frac{z-1}{z^{n-m}-1} && (13.6.17)\\
&= \frac{\overline{A}_{oq}(z)}{B_{oq}(z)(z^{n-m-1}+z^{n-m-2}+z^{n-m-3}+\ldots+z+1)} && (13.6.18)\\
T_{oq}(z) &= \frac{1}{z^{n-m}} && (13.6.19)
\end{aligned}
$$

"Minimum-time dead-beat" digital controller:

$$\alpha = \frac{1}{B_{oq}(1)} \qquad (13.6.30)$$

$$C_q(z) = \frac{\alpha A_{oq}(z)}{z^n - \alpha B_{oq}(z)} \qquad (13.6.32)$$