ELEC ENG 4CL4:

Control System Design

Notes for Lecture #27

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Discrete Delta Domain Models

The shift operator (*as described above*) is used in the vast majority of digital control and digital signal processing work. However, in some applications the shift operator can lead to difficulties. The reason for these difficulties are explained below.

Consider the first order continuous time equation

$$\rho y(t) + y(t) = \frac{dy(t)}{dt} + y(t) = u(t)$$

and the corresponding discretized shift operator equation is of the form:

$$a_2 q y(t_k) + a_1 y(t_k) = b_1 u(t_k)$$

Expanding the differential explicitly as a limiting operation, we obtain the following form of the continuous time equation:

$$\lim_{\Delta \to 0} \left(\frac{y(t+\Delta) - y(t)}{\Delta} \right) + y(t) = u(t)$$

If we now compare the discrete model to the approximate expanded form, namely

$$a_2 y(t+\Delta)+a_1 y(t)=b_1 u(t); \quad \text{where}\ \Delta=t_{k+1}-t_k$$

we then see that the fundamental difference between continuous and discrete time is that the discrete model describes absolute displacements (i.e. $y(t+\Delta)$ in terms of $y(t)$, etc.) whereas the differential equation describes the increment

$$\left(\text{i.e.}\ \frac{y(t+\Delta)-y(t)}{\Delta}\right)$$

This fundamental difficulty is avoided by use of an alternative operator; namely the *Delta operator*:

$$\boxed{\delta\left(f(k\Delta)\right) \triangleq \frac{f((k+1)\Delta) - f(k\Delta)}{\Delta}}$$

For sampled signals, an important feature of this operation is the observation that

$$\lim_{\Delta \to 0} \left[\delta\{f(k\Delta)\}\right] = \rho(f(t))$$

i.e., the Delta operator acts as a derivative in the limit as the sampling period →0. Note, however, that *no approximations* will be involved in employing the Delta operator for finite sampling periods since we will derive *exact* model descriptions relevant to this operator at the given sampling rate.

We next develop an alternative discrete transform (*which we call the Delta transform*) which is the appropriate transform to use with the Delta operator, i.e.

Time Domain	*Transfer Domain*
q δ	Z-transform delta transform

Discrete Delta Transform

We define the Discrete Delta Transform pair as:

$$\mathcal{D}\left[y(k\Delta)\right] \stackrel{\triangle}{=} Y_\delta(\gamma) = \sum_{k=0}^{\infty}(1+\gamma\Delta)^{-k}y(k\Delta)\Delta$$

$$\mathcal{D}^{-1}\left[Y_\delta(\gamma)\right] = y(k\Delta) = \frac{1}{2\pi j}\oint(1+\gamma\Delta)^{k-1}Y_\delta(\gamma)d\gamma$$

The Discrete Delta Transform can be related to Z-transform by noting that

$$Y_\delta(\gamma) = \Delta Y_q(z)\Big|_{z=\Delta\gamma+1}$$

where $Y_q(z) = Z[(k\Delta)]$. Conversely

$$Y_q(z) = \frac{1}{\Delta}Y_\delta(\gamma)\Big|_{\gamma=\frac{z-1}{\Delta}}$$

- The next slide shows a table of Delta transform pairs;
- The slide after next lists some Delta transform properties.

$f[k] \quad (k \geq 0)$	$\mathcal{D}[f[k]]$	Region of Convergence
1	$\dfrac{1+\Delta\gamma}{\gamma}$	$\left\lvert\gamma + \dfrac{1}{\Delta}\right\rvert > \dfrac{1}{\Delta}$
$\dfrac{1}{\Delta}\delta_K[k]$	1	$\lvert\gamma\rvert < \infty$
$\mu[k] - \mu[k-1]$	$\dfrac{1}{\Delta}$	$\lvert\gamma\rvert < \infty$
k	$\dfrac{1+\Delta\gamma}{\Delta\gamma^2}$	$\left\lvert\gamma + \dfrac{1}{\Delta}\right\rvert > \dfrac{1}{\Delta}$
k^2	$\dfrac{(1+\Delta\gamma)(2+\Delta\gamma)}{\Delta^2\gamma^3}$	$\left\lvert\gamma + \dfrac{1}{\Delta}\right\rvert > \dfrac{1}{\Delta}$
$e^{\alpha\Delta k} \quad \alpha \in \mathsf{C}$	$\dfrac{1+\Delta\gamma}{\gamma - \frac{e^{\alpha\Delta}-1}{\Delta}}$	$\left\lvert\gamma + \dfrac{1}{\Delta}\right\rvert > \dfrac{e^{\alpha\Delta}}{\Delta}$
$ke^{\alpha\Delta k} \quad \alpha \in \mathsf{C}$	$\dfrac{(1+\Delta\gamma)e^{\alpha\Delta}}{\Delta\left(\gamma - \frac{e^{\alpha\Delta}-1}{\Delta}\right)^2}$	$\left\lvert\gamma + \dfrac{1}{\Delta}\right\rvert > \dfrac{e^{\alpha\Delta}}{\Delta}$
$\sin(\omega_o\Delta k)$	$\dfrac{(1+\Delta\gamma)\omega_o \mathrm{sinc}(\omega_o\Delta)}{\gamma^2 + \Delta\phi(\omega_o,\Delta)\gamma + \phi(\omega_o,\Delta)}$ where $\mathrm{sinc}(\omega_o\Delta) = \dfrac{\sin(\omega_o\Delta)}{\omega_o\Delta}$ and $\phi(\omega_o,\Delta) = \dfrac{2(1-\cos(\omega_o\Delta))}{\Delta^2}$	$\left\lvert\gamma + \dfrac{1}{\Delta}\right\rvert > \dfrac{1}{\Delta}$
$\cos(\omega_o\Delta k)$	$\dfrac{(1+\Delta\gamma)(\gamma + 0.5\Delta\phi(\omega_o,\Delta))}{\gamma^2 + \Delta\phi(\omega_o,\Delta)\gamma + \phi(\omega_o,\Delta)}$	$\left\lvert\gamma + \dfrac{1}{\Delta}\right\rvert > \dfrac{1}{\Delta}$

Table 12.3: *Delta Transform Table*

$f[k]$	$\mathcal{D}[f[k]]$	Names
$\sum_{i=1}^{l} a_i f_i[k]$	$\sum_{i=1}^{l} a_i F_i(\gamma)$	Partial fractions
$f_1[k+1]$	$(\Delta\gamma+1)(F_1(\gamma)-f_1[0])$	Forward shift
$\frac{f_1[k+1]-f_1[k]}{\Delta}$	$\gamma F_1(\gamma)-(1+\gamma\Delta)f_1[0]$	Scaled difference
$\sum_{l=0}^{k-1} f[l]\Delta$	$\frac{1}{\gamma}F(\gamma)$	Reimann sum
$f[k-1]$	$(1+\gamma\Delta)^{-1}F(\gamma)+f[-1]$	Backward shift
$f[k-l]\mu[k-l]$	$(1+\gamma\Delta)^{-l}F(\gamma)$	
$kf[k]$	$-\frac{1+\gamma\Delta}{\Delta}\frac{dF(\gamma)}{d\gamma}$	
$\frac{1}{k}f[k]$	$\int_{\gamma}^{\infty}\frac{F(\zeta)}{1+\zeta\Delta}d\zeta$	
$\lim_{k\to\infty} f[k]$	$\lim_{\gamma\to 0}\gamma F(\gamma)$	Final value theorem
$\lim_{k\to 0} f[k]$	$\lim_{\gamma\to\infty}\frac{\gamma F(\gamma)}{1+\gamma\Delta}$	Initial value theorem
$\sum_{l=0}^{k-1} f_1[l]f_2[k-l]\Delta$	$F_1(\gamma)F_2(\gamma)$	Convolution
$f_1[k]f_2[k]$	$\frac{1}{2\pi j}\oint F_1(\zeta)F_2\left(\frac{\gamma-\zeta}{1+\zeta\Delta}\right)\frac{d\zeta}{1+\zeta\Delta}$	Complex convolution
$(1+a\Delta)^k f_1[k]$	$F_1\left(\frac{\gamma-a}{1+a\Delta}\right)$	

Table 12.4: *Delta Transform properties. Note that $F_i(\gamma) = D[fi[k]]$, $\mu[k]$ denotes, as usual, a unit step, $f[\infty]$ must be well defined and the convolution property holds provided that $f_1[k] = f_2[k] = 0$ for all $k < 0$.*

Why is the Delta Transform sometimes better than the Z-Transform?

As can be seen from by comparing the Z-transform given in Table 12.1 with those for the Laplace Transform given in Table 4.1, expressions in Laplace and Z-transform do not exhibit an obvious structural equivalence. Intuitively, we would expect such an equivalence to exist when the discrete sequence is obtained by sampling a continuous time signal.

We will show that this indeed happens if we use the alternative delta operator.

In particular, by comparing the entries in Table 12.3 (*The Delta Transform*) with those in Table 4.1 (*The Laplace Transform*) we see that a key property of Delta Transforms is that they converge to the associated Laplace Transform as $\Delta \rightarrow 0$, i.e.

$$\lim_{\Delta \rightarrow 0} Y_\delta(\gamma) = Y(s)\Big|_{s=\gamma}$$

We illustrate this property by a simple example:

Example 12.9

Say that $\{y[k]\}$ arises from sampling, at period Δ, a continuous time exponential $e^{\beta t}$. Then

$$y[k] = e^{\beta k \Delta}$$

and, from Table 12.3

$$Y_\delta(\gamma) = \frac{1+\gamma\Delta}{\gamma - \left[\frac{e^{\beta\Delta}-1}{\Delta}\right]}$$

In particular, note that as $\Delta \longrightarrow 0$, $Y_\delta(\gamma) \rightarrow \frac{1}{\gamma - \beta}$ which is the Laplace transform of $e^{\beta t}$.

Hence we confirm the close connections between the Delta and Laplace Transforms.

How do we use Delta Transforms?

We saw earlier in this chapter that Z-transforms could be used to convert discrete time models expressed in terms of the shift operator into algebraic equations. Similarly, the Delta Transform can be used to convert difference equations (*expressed in terms of the Delta operator*) into algebraic equations. The Delta Transform also provides a smooth transition from discrete to continuous time as the sampling rate increases.

We next examine several properties of discrete time models, beginning with the issue of stability.

Discrete System Stability

Relationship to Poles

We have seen that the response of a discrete system (in the shift operator) to an input $U(z)$ has the form

$$Y(z) = G_q(z)U(z) + \frac{f_q(z, x_o)}{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)}$$

where $\alpha_1 \ldots \alpha_n$ are the poles of the system.

We then know, via a partial fraction expansion, that $Y(z)$ can be written as

$$Y(z) = \sum_{j=1}^{n} \frac{\beta_j z}{z-\alpha_j} + \text{terms depending on } U(z)$$

where, for simplicity, we have assumed non repeated poles.

The corresponding time response is

$$y[k] = \beta_j \left[\alpha_j\right]^k + \text{ terms depending on the input}$$

Stability requires that $[\alpha_j]^k \rightarrow 0$, which is the case if $[\alpha_j] < 1$.

Hence stability requires the poles to have magnitude less than 1, i.e. to lie inside a unit circle centered at the origin.

Delta Domain Stability

We have seen that the delta domain is simply a shifted and scaled version of the Z-Domain, i.e. $\gamma = \frac{Z-1}{\Delta}$ and $Z = \gamma\Delta + 1.$ It follows that the Delta Domain stability boundary is simply a shifted and scaled version of the Z-domain stability boundary. In particular, the delta domain stability boundary is a circle of radius $^1/_\Delta$ centered on - $^1/_\Delta$ in the γ domain. Note again the close connection between the continuous s-domain and discrete δ-domain, since the δ-stability region approaches the s-stability region (OLHP) as $\Delta \to 0$.

Discrete Models for Sampled Continuous Systems

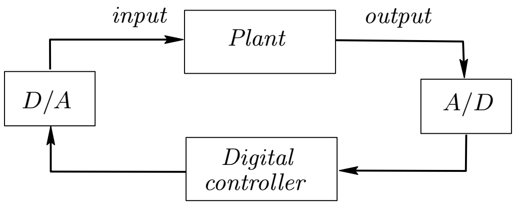
So far in this chapter, we have assumed that the model is already given in discrete form. However, often discrete models arise by sampling the output of a continuous time system. We thus next examine how to obtain discrete time models which link the sampled output of a continuous time system to a sampled input.

We are thus interested in modelling a continuous system operating under computer control.

A typical way of making this interconnection is shown on the next slide.

The analogue to digital converter (A/D in the figure) implements the process of sampling (at some fixed period Δ). The digital to analogue converter (D/A in the figure) interpolates the discrete control action into a function suitable for application to the plant input.

Figure 12.4: *Digital control of a continuous time plant*



Details of how the plant input is reconstructed

When a zero order hold is used to reconstruct $u(t)$, then

$$u(t) = u[k] \qquad \text{for} \qquad k\Delta \leq t < (k+1)\Delta$$

Note that this is the staircase signal shown earlier in Figure 12.2. Discrete time models typically relate the sampled signal $y[k]$ to the sampled input $u[k]$. Also a digital controller usually evaluates $u[k]$ based on $y[j]$ and $r[j]$, where $\{r(k\Delta)\}$ is the reference sequence and $j \leq$ k.

Using Continuous Transfer Function Models

We observe that the generation of the staircase signal $u(t)$, from the sequence $\{u(k)\}$ can be modeled as in Figure 12.5.

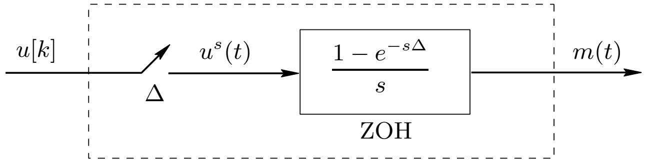
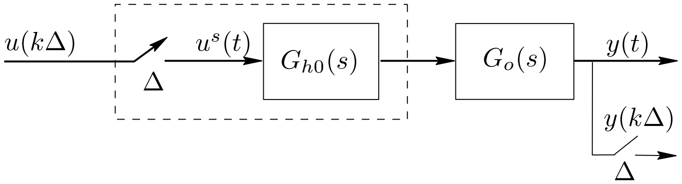


Figure 12.5: *Zero order hold*

Figure 12.6: *Discrete time equivalent model with zero order hold*

Combining the circuit on the previous slide with the plant transfer function $G_0(s)$, yields the equivalent connection between input sequence, $u(k\Delta)$, and sampled output $y(k\Delta)$ as shown below:



We saw earlier that the transfer function of a discrete time system, in Z-transform form is the Z-transform of the output (the sequence $\{y[k]\}$) when the input, $u[k]$, is a Kronecker delta, with zero initial conditions. We also have, from the previous slide, that if $u[k] = \delta_K[k]$, then the input to the continuous plant is a Dirac Delta, i.e. $u^s(t) = \delta(t)$. If we denote by $H_{eq}(z)$ the transfer function from $U_q(z)$ to $Y_q(z)$, we then have the following result.

$$
\begin{aligned}
H_{oq}(z) &= \mathcal{Z}\left[\text{the sampled impulse response of } G_{h0}(s)G_o(s)\right] \\
&= (1 - z^{-1})\mathcal{Z}\left[\text{the sampled step response of } G_o(s)\right]
\end{aligned}
$$

Example 12.10

Consider the d.c. servo motor problem used as motivation for this chapter. The continuous time transfer function is

$$G_o(s) = \frac{b_0}{s(s+a_0)}$$

Using the result on the previous slide we see that

$$\begin{aligned} H_{oq}(z) &= \frac{(z-1)}{z}\mathcal{Z}\left\{\frac{b_0}{a_0}(k\Delta) - \frac{b_0}{a_0^2} + \frac{b_0}{a_0^2}e^{-\delta k}\right\} \\ &= \frac{(z-1)}{a_0^2}\left\{\frac{a_0 b_0 z\Delta}{(z-1)^2} - \frac{b_0 z}{z-1} + \frac{b_0}{z-e^{-a_0\Delta}}\right\} \\ &= \frac{\left(b_0 a_0\Delta + b_0 e^{-a_0\Delta} - b_0\right)z - b_0 a_0\Delta e^{-a_0\Delta} - b_0 e^{-a_0\Delta} + b_0}{a_0^2(z-1)(z-e^{-a_0\Delta})} \end{aligned}$$

This model is of the form:

$$y\left(\overline{k+1}\Delta\right)=\bar{a}_1 y\left(\bar{k}\Delta\right)+\bar{a}_0 y\left(\overline{k-1}\Delta\right)+\bar{b}_1 u\left(\bar{k}\Delta\right)+\bar{b}_2 u\left(\overline{k-1}\Delta\right)$$

Note that this is a second order transfer function with a first order numerator.

The reader may care to check that this is consistent with the input-output model which was stated without proof in the introduction i.e.

$$H_{0q}(z)=\frac{\bar{b}_1 z+\bar{b}_0}{Z^2-\bar{a}_1 Z-\bar{a}_0}$$

We have thus fulfilled one promise of showing where this model comes from.