ELEC ENG 4CL4:

Control System Design

Notes for Lecture #30

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Is a Dedicated Digital Theory Really Necessary?

We could well ask if it is necessary to have a separate theory of digital control or could one simply map over a continuous design to the discrete case. Three possible design options are:

1) Design the controller in continuous time, discretize the result for implementation and ensure that the sampling constraints do not significantly affect the final performance.
2) Work in discrete time by doing an exact analysis of the *at-sample* response and ensure that the intersample response is not too surprising, or
3) carry out an exact design by optimizing the continuous response with respect to the (constrained) digital controller.

We will analyze and discuss these 3 possibilities below.

1. Approximate Continuous Designs

Given a continuous controller, $C(s)$, we mention three methods drawn from the digital signal processing literature for determining an *equivalent* digital controller.

1.1 Simply take a continuous time controller expressed in terms of the Laplace variable, s and then replace every occurrence of s by the corresponding delta domain operator γ. This leads to the following digital control law:

$$\overline{C}_1(\gamma) = C(s)\Big|_{s=\gamma}$$

where $C(s)$ is the transfer function of the continuous time controller and where $\overline{C}_1(\gamma)$ is the resultant transfer function of the discrete time controller in delta form.

1.2 Convert the controller to a zero order hold discrete equivalent. This is called a *step invariant transformation*. This leads to

$$\overline{C}_2(\gamma) = \mathcal{D}\left[\text{sampled impulse response of}\left\{C(s)G_{h0}(s)\right\}\right]$$

where $C(s)$, $G_{h0}(s)$ and $\overline{C}_2(\gamma)$ are the transfer functions of the continuous time controller, zero order hold and resultant discrete time controller respectively.

1.3 We could use a more sophisticated mapping from s to γ. For example, we could carry out the following transformation, commonly called a *bilinear transformation with pre-warping*. We first let

$$s = \frac{\alpha\gamma}{\frac{\Delta}{2}\gamma + 1} \Longleftrightarrow \gamma = \frac{s}{\alpha - \frac{\Delta}{2}s}$$

The discrete controller is then defined by

$$\overline{C}_3(\gamma) = C(s)\big|_{s=\frac{\alpha\gamma}{\frac{\Delta}{2}\gamma+1}}$$

We next choose α so as to match the frequency responses of the two controllers at some desired frequency, say ω^*. For example, one might choose ω^* as the frequency at which the continuous time sensitivity function has its maximum value.

We illustrate the above 3 ideas below for a simple system.

Example 13.2

A plant has a nominal model given by

$$G_o(s) = \frac{1}{(s-1)^2}$$

Synthesize a continuous time PID controller such that the dominant closed loop poles are the roots of the polynomial $s^2 + 3s + 4$.

The closed loop characteristic polynomial Acl(s) is chosen as

$$A_{cl}(s) = (s^2 + 3s + 4)(s^2 + 10s + 25)$$

where the factor $s^2 + 10s + 25$ has been added to ensure that the degree of $A_{cl}(s)$ is 4, which is the minimum degree required for an arbitrarily chosen $A_{cl}(s)$.

On solving the pole assignment equation we obtain $P(s) = 88s^2 + 100s + 100$ and $\bar{L}(s) = s + 15$. This leads to the following PID controller

$$C(s) = \frac{88s^2 + 100s + 100}{s(s+15)}$$

We next study the 3 procedures suggested earlier for obtaining an *equivalent* digital control law.

1.1 Method 1 - Here to obtain a discrete time PID controller we simply substitute s by γ. In this case, this yields

$$C_\delta(\gamma) = \frac{88\gamma^2 + 100\gamma + 100}{\gamma(\gamma + 15)}$$

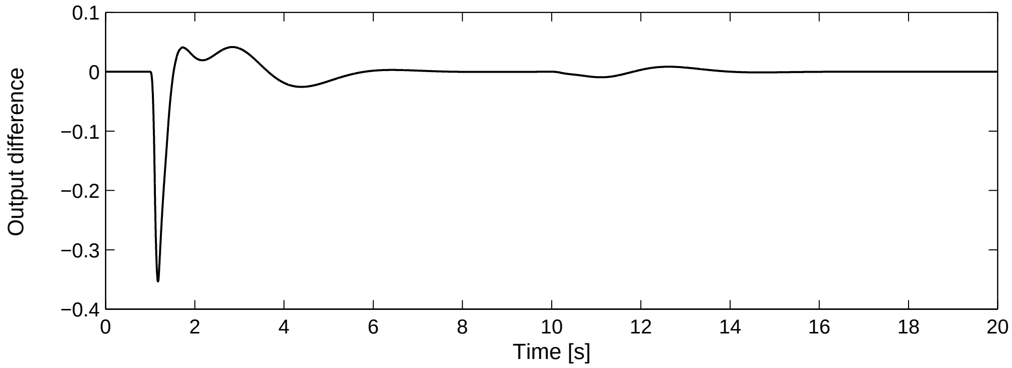
or, in Z transform form

$$C_q(z) = \frac{88z^2 - 166z + 79}{(z-1)(z+0.5)}$$

where we have assumed a sampling period $\Delta = 0.1$.

The continuous and the discrete time loops are simulated with SIMULINK for a unit step reference at $t = 1$ and a unit step input disturbance at $t = 10$. The difference of the plant outputs is shown in Figure 13.3.

Figure 13.3: *Difference in plant outputs due to discretization of the controller* (*sampling period =0.1*[*s*])



For the above example, we see that method 1.1 (*i.e. simply replace* s *by* γ) has led to an entirely satisfactory digital control law. However, this isn't always the case as we show by the next example.

Example 13.3

The system nominal transfer function is given by

$$G_o(s) = \frac{10}{s(s+1)}$$

and the continuous time controller is

$$C(s) = \frac{0.416s + 1}{0.139s + 1}$$

Replace the controller by a digital controller with $\Delta = 0.157[s]$ preceded by a sampler and followed by a ZOH using the three approximations outlined earlier.

Three methods for directly mapping a continuous controller to discrete time

1.1 Replacing s by γ in $C(s)$ we get

$$\overline{C}_1(\gamma) = \frac{0.416\gamma + 1}{0.139\gamma + 1}$$

1.2 The ZOH equivalent of $C(s)$ is

$$\overline{C}_2(\gamma) = \frac{0.694\gamma + 1}{0.232\gamma + 1}$$

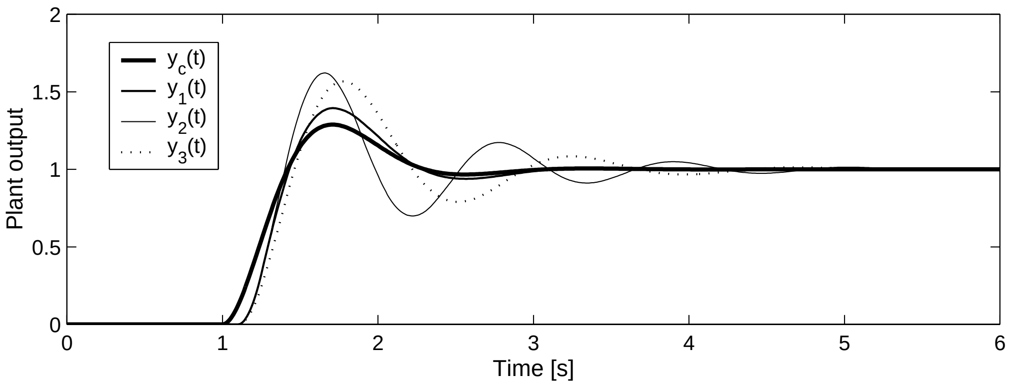
1.3 For the bilinear mapping with pre-warping, we choose $\omega^* = 5.48$. This gives $\alpha = 0.9375$ and the resulting controller becomes

$$\overline{C}_3(\gamma) = C(s)\Big|_{s=\frac{\alpha\gamma}{\frac{\Delta}{2}\gamma+1}} = \frac{0.4685\gamma + 1}{0.2088\gamma + 1}$$

Simulation Results

The above 3 digital controllers were simulated and their performance checked against the performance achieved with the original continuous controller. The results are shown on the next slide.

Figure 13.4: *Performance of different control designs: continuous time* ($y_c(t)$)*, simple substitution* ($y_1(t)$)*, step invariance* ($y_2(t)$) *and bilinear transformation* ($y_3(t)$).



We see from the figure that none of the approximations exactly reproduces the closed-loop response obtained with the continuous time controller. Actually for this example, we see that simple substitution (Method (1.1)) appears to give the best result and that there is not much to be gained by fancy methods here. However, it would be dangerous to draw general conclusions from this one example.

2. At-Sample Digital Design

The next option we explore is that of doing an exact digital control system design *for the sampled response*.

We recall that the sampled response is exactly described by appropriate discrete-time-models (expressed in either the shift or delta operators).

Time Domain Design

Any algebraic technique (*such as pole assignment*) has an immediate digital counterpart. Essentially all that is needed is to work with z (*or* γ) instead of the Laplace variable, s, and to keep in mind the different region for closed loop stability.

We illustrate below by several special digital control design methods.

Minimal Prototype

The basic idea in this control design strategy is to achieve zero error at the sample points in the minimum number of sampling periods, for step references and step output disturbances (with zero initial conditions). This implies that the complementary sensitivity must be of the form

$$T_o(z) = \frac{p(z)}{z^l}$$

Case 1:

The plant sampled transfer function, $G_{0q}(z)$ is assumed to have all its poles and zeros strictly inside the stability region. Then the controller can cancel the numerator and the denominator of $G_{0q}(z)$ and the pole assignment equation becomes

$$L_q(z)A_{oq}(z) + P_q(z)B_{oq}(z) = A_{clq}(s)$$

where

$$\begin{aligned} L_q(z) &= (z-1)B_{oq}(z)\overline{L}_q(z) \\ P_q(z) &= K_o A_{oq}(z) \\ A_{clq}(s) &= z^{n-m} B_{oq}(z) A_{oq}(z) \end{aligned}$$

Simplifying, we obtain

$$(z-1)\overline{L}_q(z) + K_o = z^{n-m}$$

This equation can now be solved for K_0 by evaluating the expression at $z = 1$. This leads to $K_0 = 1$, and to a controller and a complementary sensitivity given by

$$C_q(z) = [G_{oq}(z)]^{-1}\frac{1}{z^{n-m}-1}; \qquad \text{and} \qquad T_o(z) = \frac{1}{z^{n-m}}$$

We illustrate this case with an example.

Example 13.4

Consider a continuous time plant with transfer function

typo in book!

$$G_o(s) = \frac{10}{(s+2)(s+5)}$$

Synthesize a minimum prototype controller with sampling period $\Delta = 0.1$[s].

The sampled transfer function is given by

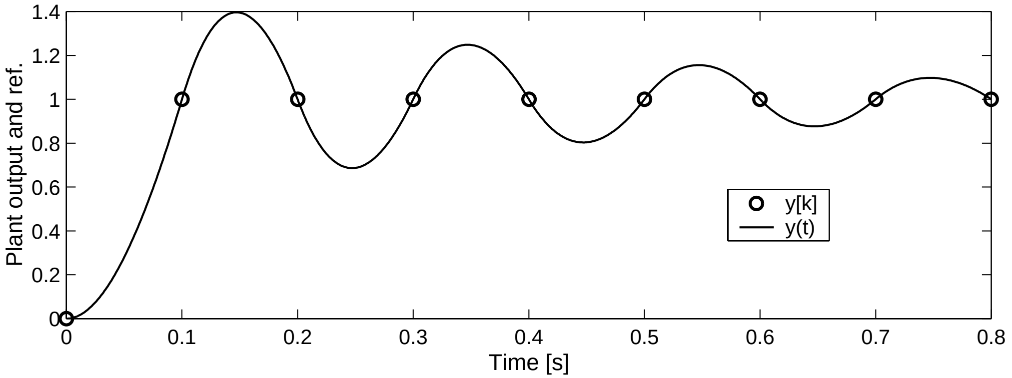
$$G_{oq}(z) = \frac{0.0398(z + 0.7919)}{(z - 0.8187)(z - 0.6065)}$$

Notice that $G_{0q}(z)$ is stable and minimum phase, with $m = 2$ and $n = 3$. The resulting minimal prototype control law is:

$$C_q(z) = \frac{25.124(z - 0.8187)(z - 0.6065)}{(z - 1)(z + 0.7919)} \qquad \text{and} \qquad T_{oq}(z) = \frac{1}{z}$$

The next slide shows a simulation of the closed loop system.

Figure 13.5: *Plant output for a unit step reference and a minimal prototype digital control.*



We see that the *sampled* response settles in exactly one sample period. This is as expected, since $T_{0q}(z) = {}^{1}/_{z}$. However, Figure 13.5 illustrates one of the weaknesses of minimal prototype control: perfect tracking is only guaranteed at the sampling instants!

(*The reader is asked to review the motivating example described in the slides for Chapter 12. Note that exactly the same problem of poor intersample response arose with the earlier example*).

Case 2:

The plant is assumed to be minimum phase and stable, except for a pole at $z = 1$, i.e. $A_{0q}(z) = (z-1)\bar{A}_{0q}(z)$. In this case, the minimal prototype idea does not require that the controller have a pole at $z = 1$. Thus, equations (13.6.6) to (13.6.8) become

$$L_q(z) = B_{oq}(z)\overline{L}_q(z)$$

$$P_q(z) = K_o\overline{A}_{oq}(z)$$

$$A_{clq}(z) = z^{n-m}B_{oq}(z)\overline{A}_{oq}(z)$$

The resulting control is as follows.

$$C_q(z) = [G_{oq}(z)]^{-1} \frac{1}{z^{n-m} - 1} = \frac{\overline{A}_{oq}(z)}{B_{oq}(z)} \frac{z-1}{z^{n-m} - 1}$$

$$= \frac{\overline{A}_{oq}(z)}{B_{oq}(z)(z^{n-m-1} + z^{n-m-2} + z^{n-m-3} + \ldots + z + 1)}$$

$$T_{oq}(z) = \frac{1}{z^{n-m}}$$

Example 13.5

Consider the servo system of Example 3.4. Recall that its transfer function is given by

$$G_o(s) = \frac{1}{s(s+1)}$$

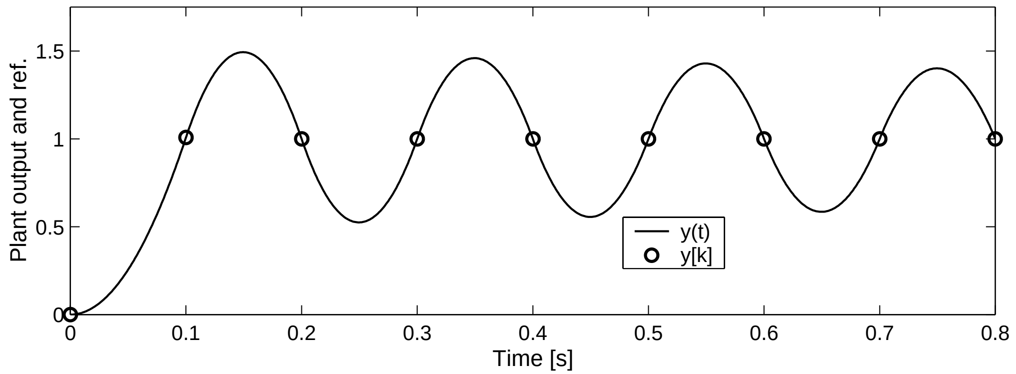
Synthesize a minimal prototype controller with sampling period $\Delta = 0.1[s]$.

$$G_{oq}(z) = 0.0048\frac{z+0.967}{(z-1)(z-0.905)}$$

$$C_q(z) = 208.33\frac{z-0.905}{z+0.967}$$

$$T_{0q}(z) = \frac{1}{z}$$

Figure 13.6: *Plant output for a unit step reference and a minimal prototype digital control. Plant with integration.*



Note that the above results are essentially identical to the simulation results presented for the motivational example given in the slides for Chapter 12.