ELEC ENG 4CL4:

Control System Design

Notes for Lecture #9

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Root Locus (RL)

Another classical tool used to study stability of equations of the type given above is root locus. The root locus approach can be used to examine the location of the roots of the characteristic polynomial as one parameter is varied.

Consider the following equation

$$1 + \lambda F(s) = 0 \qquad \text{where} \qquad F(s) = \frac{M(s)}{D(s)}$$

with $\lambda \geq 0$ and M, D have degree m, n respectively.

“Properness” of rational transfer functions

- The difference in degree between $D(s)$ and $M(s)$ is the relative degree: $n_r = n - m$.
- If $m < n$ (i.e., $n_r > 0$), we say that the transfer function is *strictly proper*.
- If $m = n$ (i.e., $n_r = 0$), we say that the transfer function is *biproper*.
- If $m \leq n$ (i.e., $n_r \geq 0$), we say that the transfer function is *proper*.
- If $m > n$ (i.e., $n_r < 0$), we say that the transfer function is *improper*.

Root locus building rules include:

R1 The number of roots of the equation $(1 + \lambda F(s) = 0)$ is equal to max{m,n}. Thus, the root locus has max{m,n} branches.

R2 From $(1 + \lambda F(s) = 0)$ we observe that s_0 belongs to the root locus (for $\lambda \geq 0$) if and only if

$$\arg F(s_0) = (2k+1)\pi \qquad \text{for} \qquad k \in \mathbb{Z}.$$

R3 From equation $(1 + \lambda F(s) = 0)$ we observe that if s_0 belongs to the root locus, the corresponding value of λ is λ_0 where

$$\lambda_0 = \frac{-1}{F(s_0)}$$

R4 A point s_0 on the real axis, i.e. $s_0 \in \mathbb{R}$, is part of the root locus (for $\lambda \geq 0$), if and only if, it is located to the left of an odd number of poles and zeros (so that R2 is satisfied).

R5 When λ is close to zero, then n of the roots are located at the poles of $F(s)$, i.e. at $p_1, p_2, \ldots, p_n$ and, if $n < m$, the other $m - n$ roots are located at ∞ (we will be more precise on this issue below).

R6 When λ is close to ∞, then m of these roots are located at the zeros of $F(s)$, i.e. at $c_1, c_2, \ldots, c_m$ and, if $n > m$, the other $n - m$ roots are located at ∞ (we will be more precise on this issue below).

R7 If $n > m$, and λ tends to ∞, then, $n - m$ roots asymptotically tend to ∞, following asymptotes which intersect at $(\sigma, 0)$, where

$$\sigma = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} c_i}{n - m}$$

The angles of these asymptotes are $\eta_1, \eta_2, \ldots, \eta_{m-n}$, where

$$\eta_k = \frac{(2k-1)\pi}{n-m}; \qquad k = 1, 2, \ldots, n - m$$

R8 If $n < m$, and λ tends to zero, then, m-n roots asymptotically tend to ∞, following asymptotes which intersect at $(\sigma, 0)$, where

$$\sigma = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} c_i}{m - n}$$

The angles of these asymptotes are $\eta_1, \eta_2, \ldots \eta_{m-n}$, where

$$\eta_k = \frac{(2k-1)\pi}{n-m}; \qquad k = 1, 2, \ldots, m-n$$

R9 When the root locus crosses the imaginary axis, say at $s = \pm j w_c$, then w_c can be computed either using the Routh Hurwitz algorithm, or using the fact that $s^2 + w_c^2$ divides exactly the polynomial $D(s) + \lambda M(s)$, for some positive real value of λ.

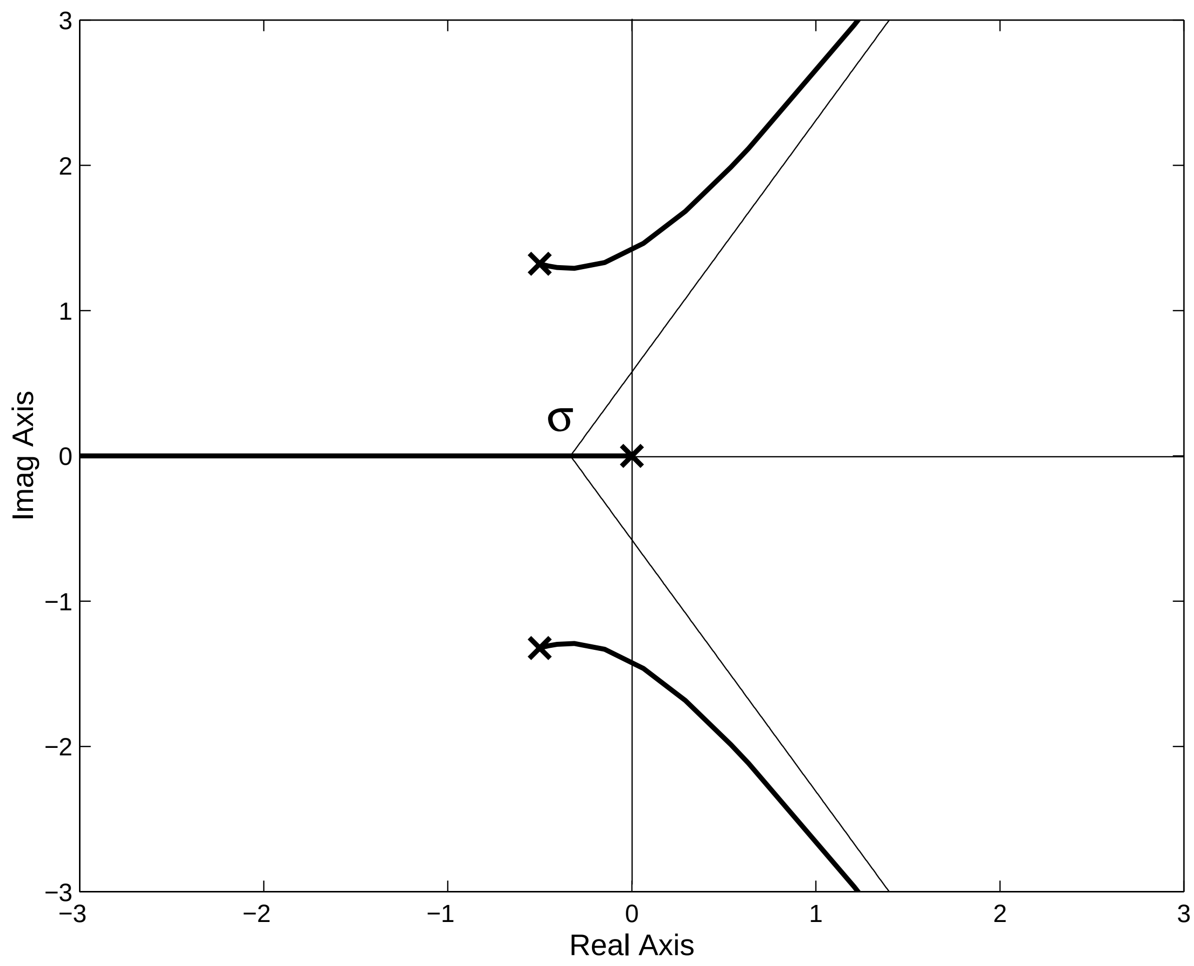
Example

Consider a plant with transfer function $G_0(s)$ and a feedback controller with transfer function $C(s)$, where

$$G_o(s) = \frac{1}{(s-1)(s+2)} \qquad \text{and} \qquad C(s) = 4\frac{s+\alpha}{s}$$

We want to know how the location of the closed loop poles change for α moving in $\mathbb{R}^+$.

Figure 5.3: *Locus for the closed loop poles when the controller zero varies*



Nominal Stability using Frequency Response

A classical and lasting tool that can be used to assess the stability of a feedback loop is Nyquist stability theory. In this approach, stability of the closed loop is predicted using the open loop frequency response of the system. This is achieved by plotting a polar diagram of the product $G_0(s)C(s)$ and then counting the number of encirclements of the (-1,0) point. We show how this works below.

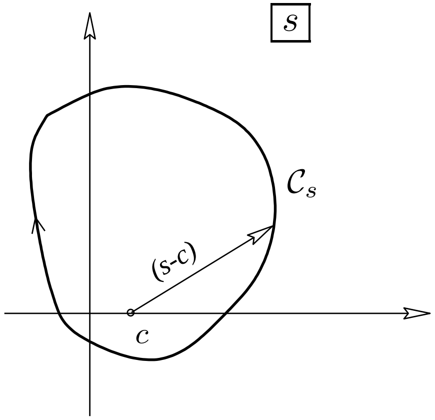
Nyquist Stability Analysis

The basic idea of Nyquist stability analysis is as follows:

assume you have a closed oriented curve C_s in $\boxed{s}$ which encircles Z zeros and P poles of the function $F(s)$. We assume that there are no poles on C_s.

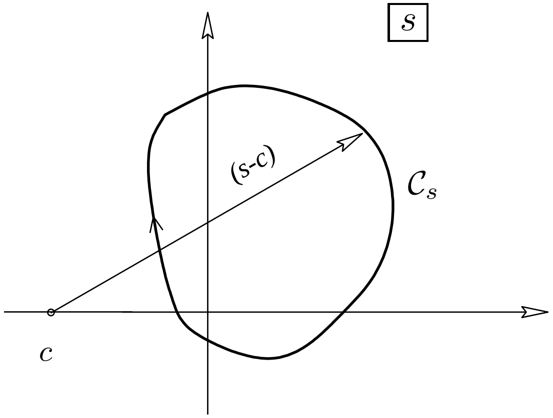
If we move along the curve C_s in a defined direction, then the function $F(s)$ maps C_s into another oriented closed curve, C_F in $\boxed{F}$.

Illustration: *Single zero function and Nyquist path* C_s *in* $\boxed{s}$



a)

c inside C_s



b)

c outside C_s

Observations

Case (a): c inside C_s

We see that as s moves clockwise along C_s, the angle of $F(s)$ changes by -2π[rad], i.e. the curve C_F will enclose the origin in $\boxed{F}$ once in the clockwise direction.

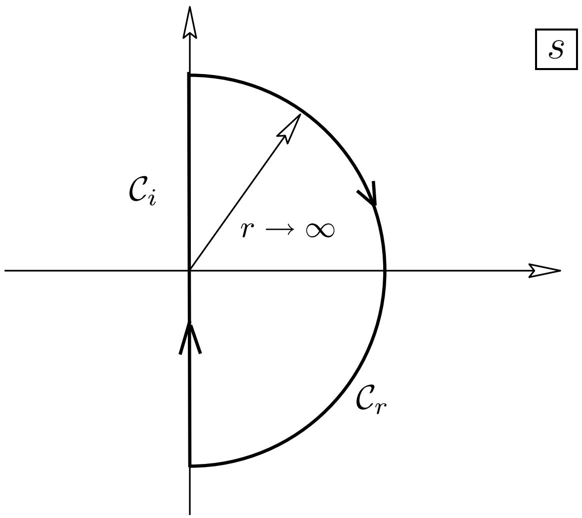
Case (b): c outside C_s

We see that as s moves clockwise along C_s, the angle of $F(s)$ changes by 0[rad], i.e. the curve C_F will not enclose the origin in $\boxed{F}$

More general result:

Consider a general function $F(s)$ and a closed curve $\mathcal{C}_s$ in $\boxed{s}$. Assume that $F(s)$ has Z zeros and P poles inside the region enclosed by $\mathcal{C}_s$. Then as s moves clockwise along $\mathcal{C}_s$, the resulting curve $\mathcal{C}_F$ encircles the origin in $\boxed{F}$ Z-P times in a clockwise direction.

To test for poles in the Right half Plane, we choose C_s as the following Nyquist path



As s traverses the Nyquist path in s, then we plot a polar plot of $F = G_0C$. Actually we shift the origin to "-1" so that encirclements of -1 count the zeros of $G_0C + 1$ in the right half plane.

Final Result

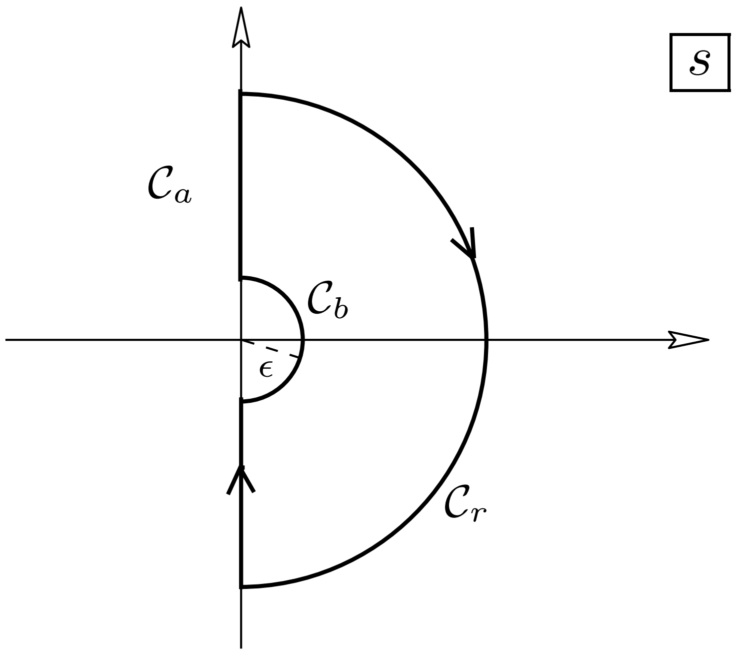
Theorem 5.1:

If a proper open loop transfer function $G_0(s)C(s)$ has P poles in the open RHP, and none on the imaginary axis, then the closed loop has Z poles in the open RHP if and only if the polar plot $G_0(s_w)C(s_w)$ encircles the point (-1,0) clockwise N=Z-P times.

Discussion

- If the system is open loop stable, then for the closed loop to be internally stable it is necessary and sufficient that no unstable cancellations occur and that the Nyquist plot of $G_0(s)C(s)$ *does not encircle the point* (-1,0).
- If the system is open loop unstable, with P poles in the open RHP, then for the closed loop to be internally stable it is necessary and sufficient that no unstable cancellations occur and that the Nyquist plot of $G_0(s)C(s)$ *encircles the point* (-1,0) *P times counterclockwise.*
- If the Nyquist plot of $G_0(s)C(s)$ passes exactly through the point (-1,0), there exists an $w_0 \in \mathbb{R}$ such that $F(jw_0) = 0$, i.e. the closed loop has poles located exactly on the imaginary axis. This situation is known as a *critical stability condition*.

Figure 5.6: *Modified Nyquist path (To account for open loop poles or zeros on the imaginary axis).*



Theorem 5.2 (Nyquist theorem):

Given a proper open loop transfer function $G_0(s)C(s)$ with P poles in the open RHP, then the closed loop has Z poles in the open RHP if and only if the plot of $G_0(s)C(s)$ encircles the point (-1,0) clockwise N=Z-P times when s travels along the modified Nyquist path.