Q. 8.3) NMP Zero at $s = z_0$

From Lemma 8.3: $\int_0^\infty e(t) e^{-z_0 t} \, dt = \frac{1}{z_0}$

$[0, \infty) = [0, t_1] \cup [t_1, \infty]$

So $\frac{1}{z_0} = \int_0^{t_1} e(t) e^{-z_0 t} \, dt + \int_{t_1}^\infty e(t) e^{-z_0 t} \, dt$

since $e(t) \leq |e(t)|$

$$\frac{1}{z_0} = \int_0^{t} e(t) e^{-z_0 t} \, dt + \int_{t_1}^\infty e(t) e^{-z_0 t} \, dt$$

$$\leq \int_0^{t_1} |e(t)| e^{-z_0 t} \, dt + \int_{t_1}^\infty |e(t)| e^{-z_0 t} \, dt$$

Since $E_{max} = \max_{t \in [0, t_1)} (e(t))$ and $|e(t)| \leq e^{-\alpha(t - t_1)}$

$$\leq \int_0^{t_1} E_{max} e^{-z_0 t} \, dt + \int_{t_1}^\infty e^{-\alpha(t - t_1)} e^{-z_0 t} \, dt$$

$$\leq E_{max} \left[-\frac{1}{z_0} e^{-z_0 t} \Big|_0^{t_1} \right] + e^{\alpha t_1} \int_{t_1}^\infty e^{-t(\alpha + z_0)} \, dt$$

$$\frac{1}{z_0} \leq E_{max} \left[-\frac{1}{z_0} e^{-z_0 t_1} + 1 \right] + \frac{e^{+\alpha t_1}}{\alpha + z_0} \left[e^{-t(\alpha + z_0)} \Big|_{t_1}^\infty \right]$$

$$\frac{1}{z_0} \leq E_{max} \left[-\frac{e^{-z_0 t_1}}{z_0} + 1 \right] + \frac{e^{-t_1 z_0}}{\alpha + z_0}$$

$$\frac{1}{z_0} - \frac{e^{-z_0 t_1}}{\alpha + z_0} \leq E_{max} \left[1 - \frac{e^{-z_0 t_1}}{z_0} \right]$$

$$\frac{\frac{1}{z_0} - \frac{e^{-z_0 t_1}}{\alpha + z_0}}{1 - \frac{e^{-z_0 t_1}}{z_0}} \leq E_{max} \quad ; \quad E_{max} \geq \frac{(\alpha + z_0) - z_0 e^{-z_0 t_1}}{(\alpha + z_0)(z_0 - e^{-z_0 t_1})}$$

Lower bound for Emax in $[0, t_1]$

Q 8.3) (b)

$$|e(t)| \leq e^{-\alpha(t-t_1)} \qquad \alpha > 0 \qquad t > t_1 > 0$$

- when $t = t_1$; $e^{-\alpha(0)} = e^{-0} = 1$, so

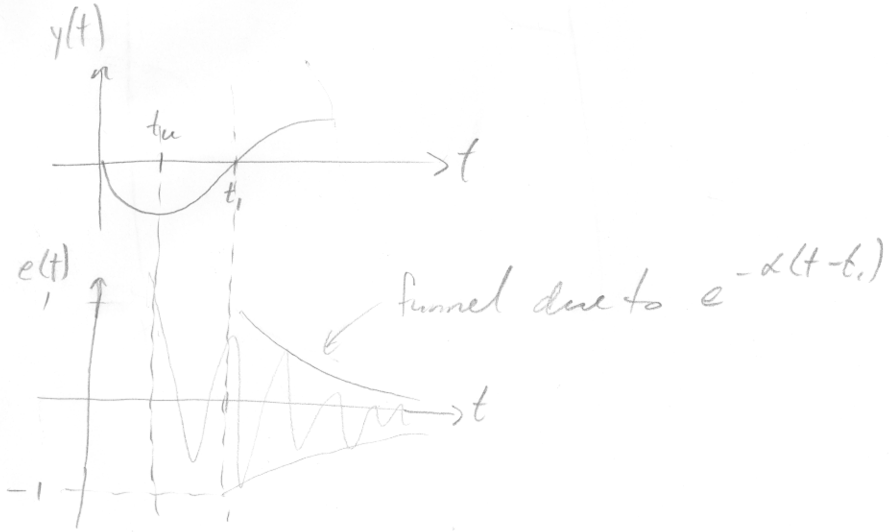
$$|e(t)| \leq 1$$

- If $|e(t)| \leq 1$ then the plant output $y(t)$ is +ve.
- A NMP zero creates an undershoot at $t = t_u$.
- So E_{max} must occur at $t = t_u$, and $|e(t_u)| = 1 + M_u > 1$

If $e^{-z_0 t_1} \approx z_0$ then E_{max} will be large,

and $e^{-z_0 t_1} \approx z_0$ at $t_1 = -\dfrac{\ln z_0}{z_0}$

i.e. when $0 < z_0 < 1$ E_{max} is large.



Problem 8.4, Goodwin et al. p 238

Given $G_o(s) = \dfrac{5(s-1)}{(s+1)(s-5)}$

This plant has to be controlled in a feedback loop with one-degree-of-freedom.

8.4.1. Determine time-domain constraints for the plant input, the plant output, and the control error in the loop. Assume exact inversion at $w=0$ and step-like reference and disturbances.

$$G_o(s) = \frac{B_o(s)}{A_o(s)} \qquad C(s) = \frac{P(s)}{L(s)}$$

Uncancelled plant poles & zeros impose algebraic / interpolation constraints on the sensitivity functions.

NB: zero @ $s=1 \rightarrow B_o(1) = 0$
pole @ $s=5 \rightarrow A_o(5) = 0$
exact inversion @ $w=0 \rightarrow L(0) = 0$ in order to get $S_o(0) = 0 \leftrightarrow T_o(0) = 1$

Interpolation constraints:

$$S_o(1) = \frac{A_o(1)\,L(1)}{A_o(1)\,L(1) + \underbrace{B_o(1)}_{0}P(1)} = 1$$

$$S_o(5) = \frac{\overbrace{A_o(5)}^{0}\,L(5)}{\underbrace{A_o(5)}_{0}\,L(5) + B_o(5)\,P(5)} = 0$$

$$T_o(1) = \frac{\overbrace{B_o(1)}^{0}\,P(1)}{A_o(1)\,L(1) + B_o(1)\,P(1)} = 0$$

$$T_0(5) = \frac{B_0(5)\,P(5)}{\cancelto{0}{A_0(5)}\,L(5) + B_0(5)\,P(5)} = 1$$

$$S_{i0}(1) = \frac{\cancelto{0}{B_0(1)}\,L(1)}{A_0(1)\,L(1) + B_0(1)\,P(1)} = 0$$

$$S_{u0}(5) = \frac{\cancelto{0}{A_0(5)}\,P(5)}{\cancelto{0}{A_0(5)}\,L(5) + B_0(5)\,P(5)} = 0$$

- Effect of unit step reference:

$$Y(s) = T_0(s)\cdot\frac{1}{s} \xrightarrow[s=1]{} \int_0^\infty y(t)e^{-t}\,dt = \lim_{s\to 1} T_0(s)\cdot\frac{1}{s} = 0\cdot 1 = 0$$

$$\xrightarrow[s=5]{} \int_0^\infty y(t)e^{-5t}\,dt = \lim_{s\to 5} T_0(s)\cdot\frac{1}{s} = 1\cdot\frac{1}{5} = \frac{1}{5}$$

$$U(s) = S_{u0}(s)\cdot\frac{1}{s} \xrightarrow[s=5]{} \int_0^\infty u(t)e^{-5t}\,dt = 0$$

$$E(s) = S_0(s)\cdot\frac{1}{s} \xrightarrow[s=1]{} \int_0^\infty e(t)e^{-t}\,dt = 1$$

$$\xrightarrow[s=5]{} \int_0^\infty e(t)e^{-5t}\,dt = 0$$

- Effect of unit step disturbance:

$$Y(s) = S_{i0}(s)\cdot\frac{1}{s} \xrightarrow[s=1]{} \int_0^\infty y(t)e^{-t}\,dt = 0$$

$$U(s) = S_0(s)\cdot\frac{1}{s} \xrightarrow[s=5]{} \int_0^\infty u(t)e^{-5t}\,dt = 0$$

$$\xrightarrow[s=1]{} \int_0^\infty u(t)e^{-t}\,dt = 1$$

$$E(s) = -S_{i0}(s)\cdot\frac{1}{s} \xrightarrow[s=1]{} \int_0^\infty e(t)e^{-t}\,dt = 0$$

- Effect of unit step disturbance:

$$Y(s) = S_0(s) \cdot \frac{1}{s} \xrightarrow[s=5]{} \int_0^{\infty} y(t) e^{-5t}\, dt = 0$$

$$\xrightarrow[s=1]{} \int_0^{\infty} y(t) e^{-t}\, dt = 1$$

$$U(s) = -S_{u0}(s) \cdot \frac{1}{s} \xrightarrow[s=5]{} \int_0^{\infty} u(t) e^{-5t}\, dt = 0$$

$$E(s) = -S_0(s) \cdot \frac{1}{s} \xrightarrow[s=5]{} \int_0^{\infty} e(t) e^{-5t}\, dt = 0$$

$$\xrightarrow[s=1]{} \int_0^{\infty} e(t) e^{-t}\, dt = 1$$

8.4.2 Why is the control of this nominal plant especially difficult?

→ contradicting requirements:

- the NMP zero sets an upper bound for the closed loop bandwidth since * indicates that if $y(t)$ settles much faster than e^{-t}, there will be a large undershoot.

- the unstable pole sets a lower bound for the closed loop bandwidth since * indicates that if $y(t)$ settles much slower than e^{-5t}, there will be a large overshoot