

Chapter 13 - Solved Problems

Solved Problem 13.1. Consider a continuous-time system having nominal model given by

$$G_o(s) = \frac{2}{(s+1)(s+2)} \quad (1)$$

Assume that this system is under digital control using a zero-order sample and hold device, with sampling period Δ . Is there a sampling period such that the sampling zero is located at $z = -1$?

Solutions to Solved Problem 13.1

Solved Problem 13.2. Use the pole assignment methodology to establish expression (13.6.32) in the book for the minimum-time dead-beat controller, $C_q(z)$.

Solutions to Solved Problem 13.2

Solved Problem 13.3. Consider a continuous-time plant having nominal model

$$G_o(s) = \frac{e^{-0.2s}}{s+1} \quad (2)$$

13.3.1 Design a minimum time dead-beat controller, assuming a sampling period $\Delta = 0.1$.

13.3.2 Determine the closed-loop polynomial, $A_{clq}(z)$.

Solutions to Solved Problem 13.3

Solved Problem 13.4. Consider a continuous-time system having a sampled transfer function given by

$$G_{oq}(z) = \frac{0.1(z+0.8)}{(z-1.5)(z-0.6)} \quad (3)$$

Design a minimum time dead-beat controller for step references

Solutions to Solved Problem 13.4

Solved Problem 13.5. A plant has a nominal model given by

$$G_o(s) = \frac{4}{(s+1)(s+4)} \quad (4)$$

13.5.1 Synthesize a continuous-time controller such that the closed-loop dynamics are dominated by poles located at $s = -2 \pm j$. It is also required that the loop exhibit zero steady state error for constant references and disturbances

13.5.2 From the result obtained above, build a digital controller using equation (13.5.1) in the book. Compare the performance of the continuous and the discrete control loops, using firstly $\Delta = 0.05$ [s] and then $\Delta = 0.4$ [s].

Solutions to Solved Problem 13.5

Chapter 13 - Solutions to Solved Problems

Solution 13.1. The plant is a second-order system, hence the general form of the sampled transfer function is

$$G_{oq}(z) \triangleq [G_o G_{h0}]_q(z) = K \frac{z - \beta}{(z - \alpha_1)(z - \alpha_2)} \quad (5)$$

One way to answer this question is to first compute $G_{oq}(z)$ using the standard procedure (see section §12.13 of the book). However, for this plant it is possible to follow a more intuitive approach, paying attention to some particular features:

- a) The discrete-time system poles are located at $z = \alpha_1 = e^{-\Delta}$ and at $z = \alpha_2 = e^{-2\Delta}$
- b) The d.c. gain in the discrete-time domain is the same as for the original continuous-time system, i.e.,

$$G_{oq}(1) = K \frac{1 - \beta}{(1 - \alpha_1)(1 - \alpha_2)} = G_o(0) = 1 \quad (6)$$

- c) If we denote by $h[k]$ the response of $G_{oq}(z)$ to a Kronecker delta, $\delta_K[k]$, and by $g(t)$ the response of $G_o(s)$ to a unit step, $\mu[k]$, where

$$g(t) = \mathcal{L}^{-1} \left[\frac{2}{(s+1)(s+2)s} \right] = \mathcal{L}^{-1} \left[\frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2} \right] = 1 - 2e^{-t} + e^{-2t} \quad (7)$$

Also $h[1] = K$. Therefore,

$$h[1] = K = g(\Delta) = 1 - 2e^{-\Delta} + e^{-2\Delta} \quad (8)$$

If we now use this expression for K in (6), we obtain

$$\beta = -e^{-\Delta} \quad (9)$$

We thus conclude that β approaches -1 when $\Delta = 0$, i.e., for an infinite sampling frequency. (See also Solved Problem 14.1)

Finally, this computation can also be carried out using a symbolic math package such as MAPLE. This can be done using the following code

```
>G(s):=2/((s+1)*(s+2)): # define the plant model;
>with(inttrans):g(t):=invlaplace(G(s)/s,s,t):# Compute the step response
>assume (Delta, positive);assume (k, integer);assume(k, positive):
>gd(k):=subs(t=k*Delta, g(t)):# sample the step response
> Goq(z):=simplify((z-1)*ztrans(gd(k),k,z)/z); # compute the sampled transfer function
```

This MAPLE code yields

$$G_{oq}(z) = \frac{ze^{\Delta} - 2ze^{2\Delta} + 1 + ze^{3\Delta} - 2e^{\Delta} + e^{2\Delta}}{(ze^{\Delta} - 1)(ze^{2\Delta} - 1)} = \frac{ze^{\Delta}(1 - 2e^{\Delta} + e^{2\Delta}) + (1 - 2e^{\Delta} + e^{2\Delta})}{(ze^{\Delta} - 1)(ze^{2\Delta} - 1)} \quad (10)$$

From where (9) is obtained.

Solution 13.2. We first assume that the pulse transfer function is given by

$$G_{oq}(z) = \frac{B_{oq}(z)}{A_{oq}(z)} \quad (11)$$

where $A_{oq}(z)$ is a *stable* n^{th} order monic polynomial¹, and $B_{oq}(z)$ is a m^{th} order polynomial, with $m < n$. Also, we assume that the controller is expressed as

$$C_q(z) = \frac{P_q(z)}{L_q(z)} \quad (12)$$

We next require that this is zero steady state error for step references and disturbances. To achieve this we put an integrator (a pole at $z = 1$) in the controller. Hence the associated Diophantine equation becomes

$$A_{oq}(z) \underbrace{(z-1)\tilde{L}_q(z)}_{L_q(z)} + B_{oq}(z)P_q(z) = A_{clq}(z) \quad (13)$$

Where the closed-loop polynomial, $A_{clq}(z)$ has degree greater or equal to $2n$ (see section §7.2 of the book). Now, say we choose $A_{clq}(z) = z^n A_{oq}(z)$, i.e., we force the cancellation of the nominal model denominator. Then, the Diophantine equation becomes

$$A_{oq}(z) \underbrace{(z-1)\tilde{L}_q(z)}_{L_q(z)} + B_{oq}(z) \underbrace{p_n A_{oq}(z)}_{P_q(z)} = z^n A_{oq}(z) \quad (14)$$

$$(z-1)\tilde{L}_q(z) + B_{oq}(z)p_n = z^n \quad (15)$$

If we evaluate at (15) at $z = 1$ we obtain

$$p_n = \frac{1}{B_{oq}(1)} \quad (16)$$

Substituting (16) into (15) yields

$$L_q(z) = (z-1)\tilde{L}_q(z) = z^n - p_n B_{oq}(z) \quad (17)$$

¹Recall that a monic polynomial is a polynomial where the leading coefficient is equal to one.

This concludes the proof, since then

$$C_q(z) = \frac{\alpha A_{oq}(z)}{z^n - \alpha B_{oq}(z)} \quad \text{where } \alpha \equiv p_n \quad (18)$$

Solution 13.3. We firstly need to compute the sampled transfer function. We observe that the pure time delay generates two poles at the origin, i.e., at $z = 0$, hence

$$G_{oq}(z) = \frac{1 - e^{-\Delta}}{z^2(z - e^{-\Delta})} = \frac{0.09516}{z^2(z - 0.9048)} = \frac{B_{oq}(z)}{A_{oq}(z)} \quad (19)$$

13.3.1 To compute the dead-beat controller one can use equation (13.6.32) from the book, with $n = 3$ and $\alpha = B_{oq}(1)^{-1} = 10.51$. We thus have

$$C_q(z) = \frac{\alpha A_{oq}(z)}{z^n - \alpha B_{oq}(z)} = \frac{10.51z^2(z - 0.9048)}{z^3 - 1} \quad (20)$$

13.3.2 The closed-loop polynomial, $A_{clq}(z)$, corresponds to the sensitivity denominator times the cancelled factors. In this case, the sensitivity denominator is z^3 , and the cancelled factor is the plant denominator $A_{oq}(z) = z^2(z - 0.9048)$. Thus

$$A_{clq}(z) = z^5(z - 0.9048) \quad (21)$$

This expression shows that the closed loop has a slow pole which is not evident from the loop response to changes in reference and/or output disturbances. (Recall that this response depends on the complementary sensitivity and sensitivity respectively, and they have only poles at the origin.) However, for input disturbances, the loop response will include a mode $(0.9048)^k$, since the pole at $z = 0.9048$ will appear in the input sensitivity.

Solution 13.4. If we apply expression (13.6.32) from the book to calculate the controller, we realize that an unstable pole-zero cancellation will arise. Thus the solution given in (13.6.32) from the book is not useful for this particular case. The correct solution requires that, when choosing the closed-loop polynomial, we avoid cancelling the plant unstable pole (see the solution to Solved Problem 13.2). Let us choose

$$A_{oq}(z) = \underbrace{(z - 1.5)}_{A^+(z)} \underbrace{(z - 0.6)}_{A^-(z)} \quad (22)$$

In this case, the closed-loop polynomial, $A_{clq}(z)$ cancels the stable factor $A^-(z)$. On the other hand, we know that the closed-loop poles can be arbitrarily chosen² if $A_{clq}(z)$ has a degree at least equal to $2n$, where n is the number of plant poles. Hence, for this particular plant, the minimum degree of $A_{clq}(z)$ is 4. We thus choose

$$A_{clq}(z) = z^3 A^-(z) = z^3(z - 0.6) \quad (23)$$

²With additional requirements to achieve zero steady state error for constant reference and disturbances.

The corresponding Diophantine equation becomes

$$\underbrace{A^+(z)A^-(z)}_{A_{oq}(z)} \underbrace{(z-1)\tilde{L}_q(z)}_{L_q(z)} + \underbrace{A^-(z)(p_a z + p_b)}_{P_q(z)} B_{oq}(z) = z^3 A^-(z) = z^3(z-0.6) \quad (24)$$

If the factor $A^-(z)$ is simplified in (24), then

$$\underbrace{(z-1.5)(z-1)}_{A^+(z)} \tilde{L}_q(z) + (p_a z + p_b) B_{oq}(z) = z^3 \quad (25)$$

We can then evaluate at $z = 1$ and $z = 1.5$ to obtain

$$(p_a + p_b) B_{oq}(1) = 1 \quad (26)$$

$$(1.5p_a + p_b) B_{oq}(1.5) = (1.5)^3 \quad (27)$$

Solving this set of equations we obtain $p_a = 18.2367$ and $p_b = -12.6812$. We finally have

$$L_q(z) = \frac{z^3 - (p_a z + p_b) B_{oq}(z)}{A^+(z)} = \frac{z^3 - 1.8237z^2 - 0.1908z + 1.0145}{(z-1.5)} = (z-1)(z-0.6763) \quad (28)$$

From where the controller is seen to be

$$C_q(z) = \frac{(18.2367z - 12.6812)(z - 0.6)}{(z - 1)(z - 0.6763)} \quad (29)$$

Solution 13.5.

13.5.1 Due to the specified dominance condition, we choose the closed-loop polynomial $A_{cl}(s)$ as the following fourth-order polynomial³

$$A_{cl}(s) = (s + 2 + j)(s + 2 - j)(s + 5)(s + 30) = s^4 + 15s^3 + 79s^2 + 175s + 150 \quad (30)$$

where the dominant poles $s = -2 \pm j$ are accompanied by faster poles located at $s = -5$ and $s = -30$.

The corresponding Diophantine equation is

$$\underbrace{(s+1)(s+4)}_{A_o(s)} \underbrace{s\tilde{L}(s)}_{L(s)} + \underbrace{4}_{B_o(s)} P(s) = s^4 + 15s^3 + 79s^2 + 175s + 150 \quad (31)$$

To solve equation 31, we use the MATLAB routine **paq** (in the book CDRM), this yields

$$L(s) = s(s + 10); \quad \text{and} \quad P(s) = 6.25s^2 + 33.75s + 37.5 \implies C(s) = \frac{6.25s^2 + 33.75s + 37.5}{s(s + 10)} \quad (32)$$

³Recall that, in order that we can arbitrarily choose the closed-loop poles, and force integration, the polynomial $A_{cl}(s)$ should have degree, at least, $2n$, for a plant with n poles.

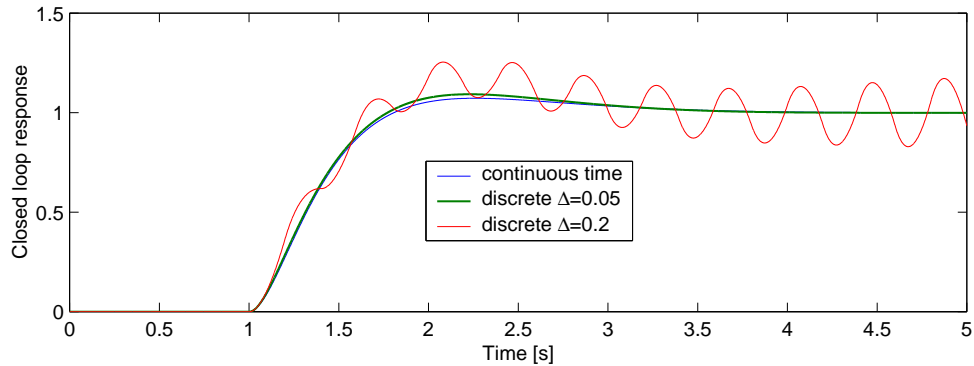


Figure 1: Closed-loop responses for a unit-step reference $r(t) = \mu(t - 1)$

13.5.2 To find a discrete-time controller in delta form, we now use equation (13.5.1) in the book,

$$C_{\delta}(\gamma) = C(s)|_{s=\gamma} = \frac{6.25\gamma^2 + 33.75\gamma + 37.5}{\gamma(\gamma + 10)} \quad (33)$$

This can be transformed to Z-form for each of the specified sampling periods (use the routine `del2z` on the book's CDROM), this yields:

$$C_{1q}(z) = \frac{6.2500z^2 - 10.8125z + 4.6563}{z^2 - 1.5z + 0.5} \quad \text{for } \Delta = 0.05 \quad (34)$$

$$C_{2q}(z) = \frac{6.25z^2 - 5.75z + 1}{z^2 - 1} \quad \text{for } \Delta = 0.2 \quad (35)$$

The performance of both digital controllers and that of the continuous-time controller are next studied using SIMULINK. A unit-step reference at $t = 1$ is applied to the three loops. The results are shown in Figure 1.

From Figure 1 one can see that for $\Delta = 0.05$ [s] the performance of the digital-control loop is very close to that of the original continuous-time design. This is a consequence of the fact that, in this case, the sampling rate which is much higher than the (continuous-time) closed-loop bandwidth. However, for the much slower sampling, i.e., $\Delta = 0.2$, the corresponding digital loop is unstable.