# Chapter 3 - Solved Problems

**Solved Problem 3.1.** *A nonlinear system has an input-output model given by*

$$\frac{dy(t)}{dt} + (1 + 0.2y(t))y(t) = u(t) + 0.2u(t)^3 \tag{1}$$

**3.1.1** *Compute the operating point(s) for $u_Q = 2$. (assume it is an equilibrium point)*

**3.1.2** *Obtain a linearized model for each of the operating points above.*

Solutions to Solved Problem 3.1

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**Solved Problem 3.2.** *A nonlinear system is described in state space form by the model*

$$\dot{x}_1(t) = -x_1(t)^2 + x_2(t) + 3u(t) \tag{2}$$
$$\dot{x}_2(t) = -2x_1(t)x_2(t) \tag{3}$$
$$y(t) = x_1(t) \tag{4}$$

*Obtain a linearized model around the equilibrium point $(u_Q, y_Q) = (2, 0)$.*

Solutions to Solved Problem 3.2

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**Solved Problem 3.3.** *Consider a discrete time system with input $u[k]$ and output $y[k]$, having an input-output model given by*

$$y[k] + 0.4y[k-1] = u[k-2] \tag{5}$$
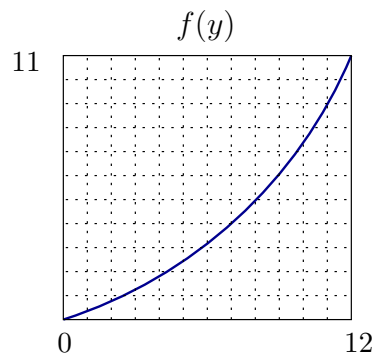
*Choose state variables and build a state space model*

Solutions to Solved Problem 3.3

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**Solved Problem 3.4.** *The input-output model for a nonlinear system is given by*

$$\frac{dy(t)}{dt} + f(y) = 2u(t) \tag{6}$$

*where $f(y)$ is the nonlinear function appearing in the figure.*
*Build a linearized model for the equilibrium point determined by $u_Q = 3$.*



Solutions to Solved Problem 3.4

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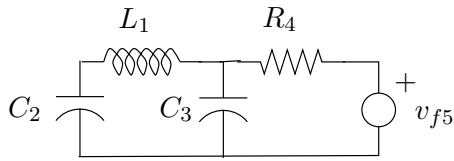**Solved Problem 3.5.** *Consider the electric network shown in Figure 1*

Figure 1: Electric network

**3.5.1** *Without using any equations, discuss how many states the system has.*

**3.5.2** *Build a state space model.*

Solutions to Solved Problem 3.5

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**Solved Problem 3.6.** *Consider a single tank of constant cross-sectional area A. The flow of water from the tank is governed by the relationship*

$$f_{out} = K\sqrt{h} \tag{7}$$

*where $h$ is the height of liquid in the tank and $K$ is a constant.*
*Assume that the flow of liquid into the tank is a control variable, $u$.*

**3.6.1** *Write down the equation governing the height of liquid in the tank.*

**3.6.2** *Linearize the model about a nominal height of $h = h^*$.*

**3.6.3** *Repeat part (i) and (ii) for a tank where the cross sectional area increases with height i.e., $A = ch$.*

Solutions to Solved Problem 3.6

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**Solved Problem 3.7.** *Consider a ball in a frictionless cone which is being rotated as shown in Figure 2. Write down the equations of motion of the ball in the vertical plane.*
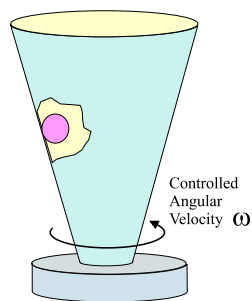


Figure 2: Cone
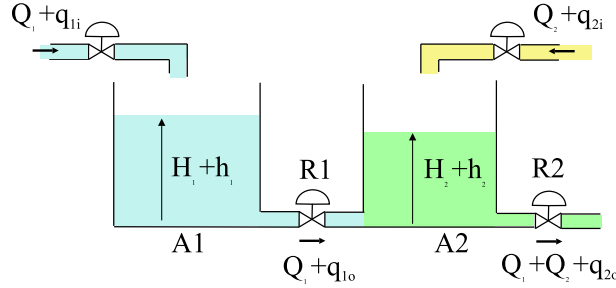
Solutions to Solved Problem 3.7

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Figure 3: Two Tanks

**Solved Problem 3.8.** *Contributed by - James Welsh, University of Newcastle, Australia.*

*Consider the two tanks system shown in Figure 3:*

$Q_1$ & $Q_2$    *are steady state flows*
$H_1$ & $H_2$    *are steady state heights (head)*
$R_1$ & $R_2$    *are value resistances*
*All lower case variables are considered to be small quantities.*

*Find a state space model for the system using $h_1$ and $h_2$ as the state variables and with $q_{1i}$ and $q_{2i}$ as the inputs.*

Solutions to Solved Problem 3.8

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**Solved Problem 3.9.** *Contributed by - Alvaro Liendo, Universidad Tecnica Federico Santa Maria, Chile.*

*Build a linear model around the equilibrium point defined by $u_Q = \sqrt{6}$ for the system:*

$$\frac{d^2 y(t)}{dt^2} + y(t)\frac{dy(t)}{dt} + y^3(t) - y(t) = 2\frac{du(t)}{dt} + u^2(t) \tag{8}$$

Solutions to Solved Problem 3.9

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**Solved Problem 3.10.** *Contributed by - Alvaro Liendo, Universidad Tecnica Federico Santa Maria, Chile.*

*Build a state space model for the system with input $u(t)$ and output $y(t)$ and having a model given by the differential equation:*

$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + y(t) = 2u(t) \tag{9}$$

Solutions to Solved Problem 3.10

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**Solved Problem 3.11.** *Contributed by - Alvaro Liendo, Universidad Tecnica Federico Santa Maria, Chile.*

*Build a state space model for the system with input $u(t)$ and output $y(t)$ and having a model given by the differential equation*

$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + y(t) = 2\frac{du(t)}{dt} \tag{10}$$

# Chapter 3 - Solutions to Solved Problems

### Solution 3.1.

**3.1.1** *The operating point $(u_Q, y_Q)$ must satisfy*

$$(1 + 0.2y_Q)y_Q = u(t) + 0.2u_Q^3 \implies 0.2y_Q^2 + y_Q - 3.6 = 0 \tag{11}$$

*This yields two operating points, $P_1$ and $P_2$ given by $(2, -7.4244)$ and $(2, 2.4244)$ respectively.*

**3.1.2** *To obtain the linearized models we can proceed in many ways. For instance, we can apply the method outlined in section §3.10. To do that we define the state as $x(t) = y(t)$. We thus have $x_Q = y_Q$ and*

$$\dot{x}(t) = f(x(t), u(t)) = -x(t) - 0.2x(t)^2 + u(t) + 0.2u(t)^3 \tag{12}$$
$$y(t) = g(x(t), u(t)) = x(t) \tag{13}$$

*If we define $x(t) = x_Q + \Delta x(t)$, $u(t) = u_Q + \Delta u(t)$, then*

$$\frac{d\Delta x(t)}{dt} = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} \Delta x(t) + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}} \Delta u(t) = (-1 - 0.4x_Q)\Delta x(t) + (1 + 0.6u_Q)\Delta u(t) \tag{14}$$
$$\Delta y(t) = \Delta x(t) \tag{15}$$

*We can also express this in input-output form as*

$$\frac{d\Delta y(t)}{dt} + (1 + 0.4y_Q)\Delta y(t) = (1 + 0.6u_Q)\Delta u(t) \tag{16}$$

*For the two operating points described above, we have*

$$P_1: \qquad \frac{d\Delta y(t)}{dt} - 1.9698\Delta y(t) = 2.2\Delta u(t) \tag{17}$$

$$P_2: \qquad \frac{d\Delta y(t)}{dt} + 1.9698\Delta y(t) = 2.2\Delta u(t) \tag{18}$$

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**Solution 3.2.** *We first need to compute the state, $(x_{1Q}, x_{2Q})$, corresponding to the equilibrium point . We notice that $x_{1Q} = y_Q = 0$, and from the first state equation we have that*

$$0 = x_{2Q} + 3u_Q \impliedby x_{2Q} = -3u_Q = 6 \tag{19}$$

*The reader can readily verify that these values also satisfy the second state equation at the equilibrium point.*

*We next express the state and plant input, output as*

$$x_1(t) = x_{1Q} + \Delta x_1(t); \quad x_2(t) = x_{2Q} + \Delta x_2(t) \tag{20}$$

$$u(t) = u_Q + \Delta u(t); \quad y(t) = y_Q + \Delta y(t) \tag{21}$$

*and we finally use the method presented in section §3.10 of the book, leading to*

$$\frac{d\Delta x_1(t)}{dt} = -2x_{1Q}\Delta x_1(t) + \Delta x_2(t) + 3\Delta u(t) = \Delta x_2(t) + 3\Delta u(t) \tag{22}$$

$$\frac{d\Delta x_2(t)}{dt} = -2x_{2Q}\Delta x_1(t) - 2x_{1Q}\Delta x_2(t) = -12\Delta x_1(t) \tag{23}$$

$$\Delta y(t) = \Delta x_1(t) \tag{24}$$

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**Solution 3.3.** *The interesting aspect of this problem is the two unit delay on the input.*

*The state variables must include all information we require to know at time $k = k_o$ such that, given the input $u[k]$, for all $k \geq k_o$, we are able to compute $y[k]$, for all $k > k_o$. From the system equation we have*

$$y[k_o + 1] = u[k_o - 1] - 0.4y[k_o] \tag{25}$$

$$y[k_o + 2] = u[k_o] - 0.4y[k_o + 1] \tag{26}$$

*We see that we require to know $y[k_o]$, and $u[k_o - 1]$ to predict the future response. We thus choose*

$$x_1[k] = y[k]; \quad x_2[k] = u[k - 1]; \tag{27}$$

*and we notice that*

$$x_1[k + 1] = y[k + 1] = u[k - 1] - 0.4y[k] = x_2[k] - 0.4x_1[k] \tag{28}$$

$$x_2[k + 1] = u[k] \tag{29}$$

$$y[k] = x_1[k] \tag{30}$$

*Setting the above in matrix form, we finally obtain*

$$\begin{bmatrix} x_1[k+1] \\ x_2[k+1] \end{bmatrix} = \begin{bmatrix} -0.4 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[k] \tag{31}$$

$$y[k] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} \tag{32}$$

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**Solution 3.4.** *The linearized model has the form*

$$\frac{d\Delta y(t)}{dt} + \left. \frac{df(y)}{dy} \right|_Q \Delta y(t) = 2\Delta u(t) \tag{33}$$

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where $y(t) = y_Q + \Delta y(t)$ and $u(t) = u_Q + \Delta u(t)$ We thus need to compute the derivative of $f(y)$ at the equilibrium point. This is done graphically as shown below
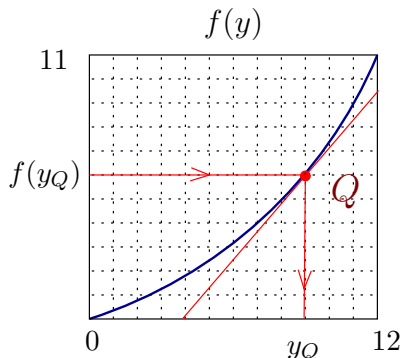


We note that, at an equilibrium point, $f(y_Q) = 2u_Q$. Thus, in our particular example, $f(y_Q) = 6$. With this value we find the operating point. The slope of the tangent at that point is the derivative of $f(y)$ at $Q$, and is approximately equal to 1.19. Hence the linearized model is given by

$$\frac{d\Delta y(t)}{dt} + 1.19\Delta y(t) = 2\Delta u(t) \tag{34}$$

## Solution 3.5.

**3.5.1** *The number of states required for the system is equal to the number of initial conditions we can arbitrarily set in the network. In this case, the number is three: an initial current in inductor $L_1$ and the initial voltages on $C_1$ and $C_2$. The reader may note that the answer can be different if we allow some singular cases. For instance, if we make $R_4 = 0$, then the number of states required will be only two, since then the voltage in $C_3$ will always be equal to the source voltage and hence it can not be set arbitrarily.*

**3.5.2** *To build the state space model we choose as state variables the electric signals $i_1(t)$, $v_2(t)$ and $v_3(t)$ which are shown in the network schematic shown in Figure 4.*
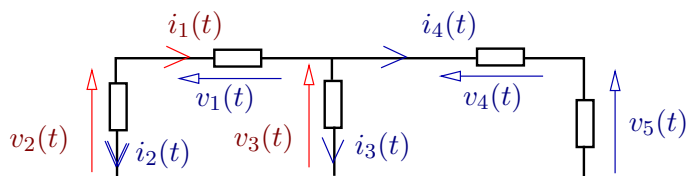


Figure 4: Electric network skeleton

*Applying Kirchoff's laws and component laws we obtain*

$$v_5(t) = v_{f5}(t) = -v_4(t) + v_3(t) = -R_4 i_4(t) + v_3(t) \tag{35}$$

$$i_4(t) = i_1(t) - i_3(t) = i_1(t) - C_3 \frac{dv_3(t)}{dt} \tag{36}$$

$$v_2(t) = v_1(t) + v_3(t) = L_1 \frac{di_1(t)}{dt} + v_3(t) \tag{37}$$

$$i_1(t) = -i_2(t) = -C_2 \frac{dv_2(t)}{dt} \tag{38}$$

*We also notice that the system input is the voltage of the independent voltage source, $v_{f5}(t)$. Rearranging the previous equations we finally obtain*

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$$\frac{di_1(t)}{dt} = \frac{1}{L_1}v_2(t) - \frac{1}{L_1}v_3(t) \tag{39}$$

$$\frac{dv_2(t)}{dt} = -\frac{1}{C_2}i_1(t) \tag{40}$$

$$\frac{dv_3(t)}{dt} = -\frac{1}{C_3}i_1(t) - \frac{1}{R_4C_3}v_3(t) + \frac{1}{R_4C_3}v_{f5}(t) \tag{41}$$

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### Solution 3.6.

**3.6.1** *The volume of liquid in the tank is*

$$V = Ah \tag{42}$$

*The rate of change of liquid in the tank is*

$$\frac{dV}{dt} = u - f_{out} \tag{43}$$

$$A\frac{dh}{dt} = u - K\sqrt{h} \tag{44}$$

$$or \quad \frac{dh}{dt} = -\frac{K}{A}\sqrt{h} - \frac{1}{A}u \tag{45}$$

**3.6.2** *If $h = h^*$, then $u = K\sqrt{h^*}$ for a steady height.*
*Let $\tilde{h} = h - h^*$, $\tilde{u} = u - u^*$.*
*Then linearizing (44) we obtain*

$$A\frac{d(h^* + \tilde{h})}{dt} = u^* + \tilde{u} - K\sqrt{h^* + \tilde{h}} \tag{46}$$

$$\simeq u^* + \tilde{u} - K\sqrt{h^*} - \frac{K}{2}(h^*)^{\frac{1}{2}}\tilde{h} \tag{47}$$

*Hence*

$$\frac{d}{dt}\tilde{h} \simeq -\frac{K}{A2\sqrt{h^*}}\tilde{h} - \frac{1}{A}\tilde{u} \tag{48}$$

**3.6.3** *Here*

$$V = Ah = ch^2 \tag{49}$$

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*Hence*

$$\frac{dV}{dt} = u - f_{out} \tag{50}$$

$$c\frac{dh^2}{dt} = u - K\sqrt{h} \tag{51}$$

$$2ch\frac{dh}{dt} = u - K\sqrt{h} \tag{52}$$

$$\frac{dh}{dt} = -\frac{K}{2c\sqrt{h}} + \frac{u}{2ch} \tag{53}$$

*Linearizing about $h^*$ gives*

$$u^* = K\sqrt{h^*} \qquad \text{as before} \tag{54}$$

*(Note that this is reasonable since we have to balance the outflow by the inflow).*

*Also*

$$\frac{d\tilde{h}}{dt} \simeq -\frac{K}{2c\sqrt{h^*}} + \frac{K}{4c}(h^*)^{-\frac{3}{2}}\tilde{h} - \frac{u^*}{2ch^*} - \frac{\tilde{u}}{2ch^*} - \frac{u^*}{2c}(h^*)^{-2}\tilde{h} \tag{55}$$

$$= \left[\frac{K}{4c}(h^*)^{-\frac{3}{2}} - \frac{u^*}{2c}(h^*)^{-2}\right]\tilde{h} - \frac{\tilde{u}}{2ch^*} \tag{56}$$

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**Solution 3.7.** *We assume that the cone makes an angle $\theta$ with the horizontal plane. Also, assume that the diameter of the base of the cone is $d_o$. Also let $h$ denote the height of the ball.*

*Then resolving the forces on the ball tangential to the wall, we have*

$$mg\cos\theta = m\left\{cot[h + \frac{d_o}{2}\tan\theta]\right\}\omega^2\sin\theta \tag{57}$$

*Differentiation with respect to time gives*

$$-csc[[h + \frac{d_o}{2}\tan\theta]^2\omega^2\sin\theta\dot{h} + cot[h + \frac{d_o}{2}\tan\theta]2\omega\dot{\omega}\sin\theta = 0 \tag{58}$$

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**Solution 3.8.** *Because we are only interested in small variations, we can assume that flow through the valves is linearly related to the difference in head.*

*The equations for Tank 1 then become:*

$$A_1\frac{dh_1}{dt} = q_{1i} - q_{1o} \tag{59}$$

$$q_{1o} = \frac{h_1 - h_2}{R_1} \tag{60}$$

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*The corresponding equations for Tank 2 are*

$$A_2 \frac{dh_2}{dt} = q_{1o} + q_{2i} - q_{2o} \tag{61}$$

$$q_{2o} = \frac{h_2}{R_2} \tag{62}$$

*Substituting 60 into 59*

$$\frac{dh_1}{dt} = \frac{1}{A_1} \left( q_{1i} - \frac{h_1 - h_2}{R_1} \right) \tag{63}$$

*Substituting 60 and 62 into 61*

$$\frac{dh_2}{dt} = \frac{1}{A_2} \left( \frac{h_1 - h_2}{R_1} + q_{2i} - \frac{h_2}{R_2} \right) \tag{64}$$

*We now define the state variables as*

$$x_1 = h_1 \tag{65}$$
$$x_2 = h_2 \tag{66}$$

*We then write 63 and 64 respectively as*

$$\dot{x}_1 = -\frac{1}{R_1 A_1} x_1 + \frac{1}{R_1 A_1} x_2 + \frac{1}{A_1} q_{1i} \tag{67}$$

$$\dot{x}_2 = -\frac{1}{R_1 A_2} x_1 - \left( \frac{1}{R_1 A_2} + \frac{1}{R_2 A_2} \right) x_2 + \frac{1}{A_2} q_{2i} \tag{68}$$

*In summary, the state space model is*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1 A_1} & \frac{1}{R_1 A_1} \\ \frac{1}{R_1 A_2} & -\left( \frac{1}{R_1 A_2} + \frac{1}{R_2 A_2} \right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{A_1} & 0 \\ 0 & \frac{1}{A_2} \end{bmatrix} \begin{bmatrix} q_{1i} \\ q_{2i} \end{bmatrix} \tag{69}$$

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{70}$$

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**Solution** **3.9.** *At equilibrium $\frac{d^n y(t)}{dt^n}$ and $\frac{d^n y(t)}{dt^n}$ are zero. Hence at the equilibrium point, the differential equation reduces to*

$$y_Q^3 - y_Q = u_Q^2 \tag{71}$$

*This leads to $(u_Q, y_Q) = (\sqrt{6}, -2)$. The other 2 solutions of the third degree equation are complex numbers.*

*The linearized model is obtained using a first order Taylor series approximation , which is as follows:*

$$f(x(t), y(t)) \approx f(x_Q, y_Q) + \frac{\partial f}{\partial x}\bigg|_{\substack{x=x_Q \\ y=y_Q}} \Delta x + \frac{\partial f}{\partial y}\bigg|_{\substack{x=x_Q \\ y=y_Q}} \Delta y \tag{72}$$

*Considering $u(t) = u_Q + \Delta u(t)$ and $y(t) = y_Q + \Delta y(t)$, we obtain*

$$y(t)\frac{dy(t)}{dt} \approx y_Q\frac{d\Delta y(t)}{dt} \tag{73}$$

$$y^3(t) \approx 3y_Q^2 \Delta y(t) + y_Q^3 \tag{74}$$

$$u^2(t) \approx 2u_Q \Delta u(t) + u_Q^2 \tag{75}$$

*Thus*

$$\frac{d^2 \Delta y(t)}{dt^2} + y_Q\frac{d\Delta y(t)}{dt} + 3y_Q^2 \Delta y(t) + y_Q^3 - \Delta y(t) - y_Q = 2\frac{d\Delta u(t)}{dt} + 2u_Q \Delta u(t) + u_Q^2 \tag{76}$$

*Using equation (71) and the value of $u_Q$ and $y_Q$ the linear model is*

$$\frac{d^2 \Delta y(t)}{dt^2} - 2\frac{d\Delta y(t)}{dt} + 12\Delta y(t) - \Delta y(t) = 2\frac{d\Delta u(t)}{dt} + 2\sqrt{6}\Delta u(t) \tag{77}$$

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**Solution 3.10.** *Consider the states $x_1(t) = \frac{dy(t)}{dt}$ and $x_2(t) = y(t)$. We then have the differential equation*

$$\dot{x}_1(t) + 3x_1(t) + x_2(t) = 2u(t) \tag{78}$$

*Thus a suitable state space model is given by*

$$\dot{x}_1(t) = -3x_1(t) - x_2(t) + 2u(t) \tag{79}$$

$$\dot{x}_2(t) = x_1(t) \tag{80}$$

$$y(t) = x_2(t) \tag{81}$$

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**Solution 3.11.** *Consider the states $\dot{x}_2(t) = x_1(t)$ and $x_1(t) = y(t)$ With these states, the differential equation can be written*

$$\ddot{x}_1(t) + 3\dot{x}_1(t) + x_1(t) = 2\dot{u}(t) \implies \ddot{x}_1(t) + 3\dot{x}_1(t) + \dot{x}_2(t) = 2\dot{u}(t)$$
$$\implies \dot{x}_1(t) + 3x_1(t) + x_2(t) = 2u(t) \tag{82}$$

*Thus a suitable state space model is*

$$\dot{x}_1(t) = -3x_1(t) - x_2(t) + 2u(t) \tag{83}$$

$$\dot{x}_2(t) = x_1(t) \tag{84}$$

$$y(t) = x_1(t) \tag{85}$$

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