Chapter 5 - Solved Problems

Solved Problem 5.1. *Show that the Nyquist Plot of $G(s) = \frac{1}{s+a}$ is a semicircle of radius $\frac{1}{2a}$ and centre $(\frac{1}{2a}, 0)$.*

   Solutions to Solved Problem 5.1

Solved Problem 5.2. *Contributed by - James Welsh, University of Newcastle, Australia.*
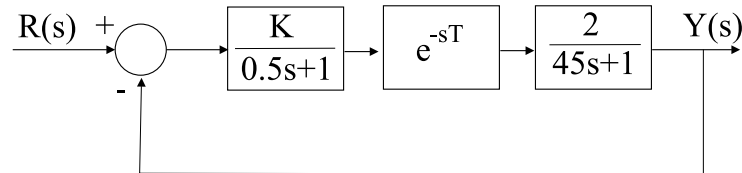

Figure 1: Level Control System

   Consider the level control system shown in Figure 1. Usually flow in a pipe does not introduce a delay since the output responds when the input changes. However, let us assume that this particular pipe exhibits characteristics of a pure time delay (similar to a conveyor belt). Thus, the delay (T) is directly proportional to the distance (d) the actuator is positioned from the tank, $(T \propto d)$.
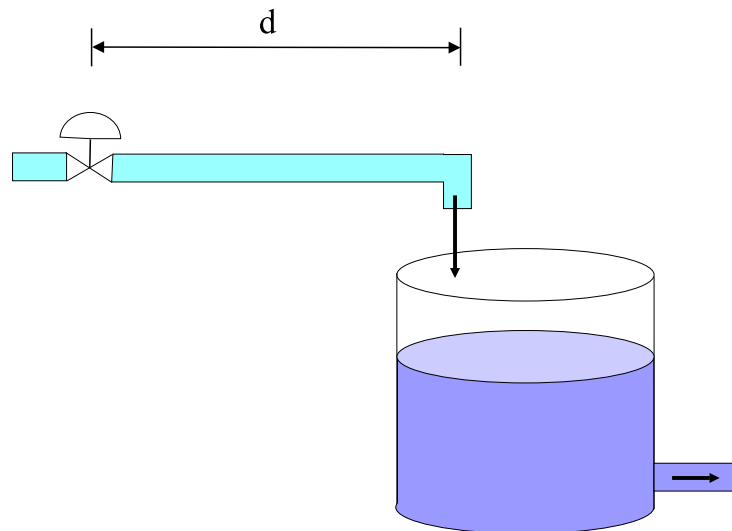

Figure 2: Actuator Position Relative to Tank

   During the design phase of the system the actuator was to be located 2m from the tank, giving a delay of 2 seconds.
   The gain margin and phase margins are required to be greater than 6dB and 50° respectively.
   To satisfy these requirements and allow for undermodelling the gain of the actuator (K) was found to be 5. Leading to (Gm=9.6dB), $(pm = 64^o)$.

However, it was discovered during construction that the actuator could not be positioned any closer than 3m.

Will the requirements for the gain and phase margins still be satisfied.

Solutions to Solved Problem 5.2

Solved Problem 5.3. *Contributed by - James Welsh, University of Newcastle, Australia.*
A unity feedback system has an open loop transfer function of

$$G(s) = \frac{Ke^{-0.5s}}{s+1} \tag{1}$$

Analytically determine the critical value of K for stability and verify by examining the Nyquist plot.

Solutions to Solved Problem 5.3

Solved Problem 5.4. *Use a Root Locus argument to show that any system having a pole on the positive real axis with a positive real zero on either side, requires an unstable controller to stabilize it.*

Solutions to Solved Problem 5.4

Solved Problem 5.5. *In a feedback control loop*

$$G_o(s) = \frac{s-2}{(s-1)(s+4)} \qquad\qquad C(s) = K\frac{s+1}{s} \tag{2}$$

*Determine $K \in \Re$, **if exists**, such that the control loop is stable.*

Solutions to Solved Problem 5.5

Solved Problem 5.6. *A plant has a nominal model given by*

$$G_o(s) = \frac{e^{-2s}}{s+1} \tag{3}$$

Assume that this plant is to be controlled in a feedback loop with a proportional controller, i.e., $C(s) = K$, $K > 0$.
Compute the critical value of K, i.e., the value which takes the loop to the verge of instability.

Solutions to Solved Problem 5.6

Solved Problem 5.7. *A feedback control is designed to achieve stability margins given by $M_g = 10[dB]$ and $M_f = \pi/4$ [rad]. When the controller is built, by error its gain is amplified by a factor 2. Do the margins change? Will the system become unstable?*

Solutions to Solved Problem 5.7

Solved Problem 5.8. *In a control loop we have that*

$$G_o(s) = \frac{1}{s(s+5)}; \quad C(s) = K\frac{s+a}{s+b} \tag{4}$$

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5.8.1 *Find K, a and b such that the complementary sensitivity, $T_o(s)$ corresponds to a canonical second order system with $\omega_n = 6$ and $\psi = 0.6$ (see section §4.8 of the book).*

5.8.2 *Assume that the reference is $r(t) = 5$ and that there is an input disturbance given by*

$$d_i(t) = 3e^{-2t} - 3 \tag{5}$$

With the controller designed above, compute the steady state value for every signal in the loop.

Solutions to Solved Problem 5.8

Solved Problem 5.9. *In a feedback control loop for a stable plant, with reference $r(t) = R_o \cos(4t)$, the controller is designed to achieve a nominal complementary sensitivity given by*

$$T_o(s) = \frac{400}{s^2 + 30s + 400} \tag{6}$$

5.9.1 *Compute the nominal steady state control error.*

5.9.2 *Assume that the design was based on a nominal model with associated multiplicative error satisfying*

$$|G_\Delta(j\omega)| \leq \rho(\omega) = \frac{\omega}{\sqrt{\omega^2 + 9}} \tag{7}$$

If the controller is used to control the (stable) calibration model, what will be the worst case regarding the amplitude of steady state control error?

Solutions to Solved Problem 5.9

Solved Problem 5.10. *In a feedback control loop the open loop transfer function $L(s) = G_o(s)C(s)$ is given by*

$$L(s) = \frac{-0.5s + 0.5}{s(s^2 + 0.4s + 4)} \tag{8}$$

5.10.1 *Draw the Nyquist plot and analyze the stability of the closed loop.*

5.10.2 *Compute the stability margins from the Nyquist plot.*

5.10.3 *Show that the sensibility peak is smaller than 4.*

Solutions to Solved Problem 5.10

Solved Problem 5.11. *In a feedback control loop of a stable and minimum phase plant, the reference is constant. However there is also an input disturbance which is a sine wave of frequency ω_o $[rad/s]$.*

5.11.1 *Determine the conditions that the controller must satisfy to achieve zero steady state error.*

5.11.2 *If the model used to design the controller is inaccurate, will we still achieve zero steady state error?*

Solutions to Solved Problem 5.11

Chapter 5 - Solutions to Solved Problems

Solution 5.1.

$$G(j\omega) = \frac{1}{j\omega + a} = \frac{a - j\omega}{a^2 + \omega^2} \tag{9}$$

Let

$$x_\omega = \Re\{G(j\omega)\} = \frac{a}{a^2 + \omega^2} \tag{10}$$

$$y_\omega = \Im\{G(j\omega)\} = \frac{-\omega}{a^2 + \omega^2} \tag{11}$$

Then

$$\left[x_\omega - \frac{1}{2a}\right]^2 + y_\omega^2 = \left[\frac{a}{a^2 + \omega^2} - \frac{1}{2a}\right]^2 + \frac{\omega^2}{[\omega^2 + a^2]^2} \tag{12}$$

$$= \frac{1}{4a^2} \tag{13}$$

We see this is the equation of a circle as required.

Solution 5.2. *We use the following MATLAB code:*

```
≫ s=tf('s')

≫ % Transfer function of tank
≫ G=2/(45*S+1);

≫ % Transfer function of actuator
≫ A=5/(0.5*s+1);

≫ % Open loop transfer function
≫ GA=G*A;

≫ % Calculate bode response
≫ [g p w]=bode(GA);
≫ g=g(:);
≫ p=p(:);

≫ % Time Delay
≫ T=3;
≫ % Calculate the phase contribution of the delay
≫ delay_angle = unwrap (angle(exp(-j.*w*T)))*180 / pi;

≫ % Add this to the system delay
≫ pd = p + delay_angle;

≫ % Determine the gain and phase margins
≫ [GM PM] = margin(g,pd,w);
```

```
≫ GM_db = 20*log10(GM)
≫ PM_deg = PM
```

Using this code, it turns out that GM = 6.5dB and PM = 51 degrees. Thus, the answer to the question is yes.

Solution 5.3. *We can rewrite the system by setting $s = jw$*

$$G(j\omega) = \frac{Ke^{-0.5j\omega}}{j\omega + 1} \tag{14}$$

$$= \frac{K(\cos(0.5\omega) - j\sin(0.5\omega))(1 - j\omega)}{1 + \omega^2} \tag{15}$$

$$= \frac{K}{1 + \omega^2} \left[\cos(0.5\omega) - \omega\sin(0.5\omega) - j\left(\sin(0.5\omega) + \omega\cos(0.5\omega)\right)\right] \tag{16}$$

The Nyquist plot crosses the imaginary axis when the imaginary part of $G(j\omega) = 0$.
Hence

$$\sin(0.5\omega) + \omega\cos(0.5\omega) = 0 \tag{17}$$

We then need to solve

$$\omega = -\tan(0.5\omega) \tag{18}$$

for the smallest positive value of ω.
   This can be done graphically using MATLAB by:

```
≫ w = logspace (-1, 1, 1000);
≫ plot (w, w + tan (0.5*w));
```

then locating the zero crossing for the smallest positive value of ω we find

$$\omega = 3.6732 \quad \text{rad/sec} \tag{19}$$

Now substituting $\omega = 3.6732$ into $G(j\omega)$.

$$G(j3.6732) = \frac{K}{1 + 3.6732^2} \left[\cos(1.8816) - 3.6732\sin(1.8816)\right] \tag{20}$$

$$= -0.2624\,K \tag{21}$$

Now the critical value occurs when $G(j\omega) = -1$.
Therefore

$$K = \frac{1}{0.2624} \tag{22}$$

$$= 3.8107 \tag{23}$$

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To verify this with the Nyquist plot we use MATLAB.

```
≫ w = logspace (-2, 2, 1000);
≫ Gjw = 3.8107*exp (-0.5*j.*w)./(j.*w + 1);
≫ plot (real (Gjw), imag (Gjw));
≫ grid on;
```
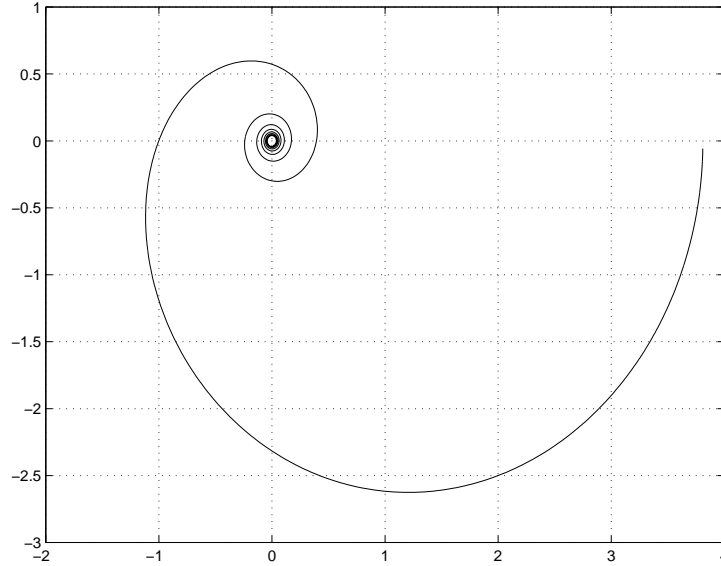

Figure 3: Nyquist Plot

The result is shown in Figure 3. As expected the plot passes through the '-1' point.

Solution 5.4. *Whenever you have a pole between two zeros on the real axis, then the root locus must lie on the real axis between the pole and one of the zeros for either positive or negative feedback gain. Thus, the only option to stabilize the system is to place a second real pole (in the controller) between the two zeros.*

Solution 5.5. *The closed loop characteristic polynomial is given by the numerator of $1 + G_o(s)C(s)$, i.e.*

$$p(s) = s^3 + (3 + K)s^2 - (K + 4)s - 2K \tag{24}$$

On applying the Routh criterion (see subsection §5.5.3 of the book) we obtain

$$
\begin{array}{c|cc}
s^3 & 1 & -(K+4) \\[2mm]
s^2 & 3+K & -2K \\[2mm]
s^1 & -\dfrac{K^2+5K+12}{3+K} & \\[4mm]
s^0 & -2K &
\end{array}
\tag{25}
$$

From the array above we observe that $p(s)$ has all its roots in the open LHP if and only if the following three conditions are simultaneously satisfied

$$
3+K > 0 \tag{26}
$$
$$
K^2 + 5K + 12 < 0 \tag{27}
$$
$$
-2K > 0 \tag{28}
$$

The first and third conditions require that $-3 < K < 0$. On the other hand, the second condition can be re-written as $(K+2.5)^2+5.75 < 0$. This form allows us to see that the condition cannot be met by any real value of K. In summary, there is no real value for K which stabilizes the closed loop.

Solution 5.6. *Note that this problem is similar to Solved Problem 5.3. Here we will use a slightly different (but equivalent) solution procedure. We define the function*

$$
F(s) = \frac{G_o(s)C(s)}{K} = G_o(s) = \frac{e^{-2s}}{s+1} \tag{29}
$$

The Bode diagrams for $F(s)$ are shown in Figure 4.

Since we can only alter the Bode diagrams by multiplying the open loop transfer function by a positive constant, we are able to shift vertically the magnitude diagram. Hence, the critical condition will arise when K is chosen to take the magnitude plot to 0 [dB] at the frequency at which the phase is $-\pi$ [rad]. This is shown by the vertical arrow on Figure 4. From there we estimate 4 [dB] as the maximum gain we can add. This is equivalent to a critical gain equal to $10^{0.2}$.

*We also have that the critical gain is equal to the Gain Margin M_g. We could thus also use the MATLAB command **margin**.*

Solution 5.7. *From the definitions of stability margins (see section §5.8 of the book), we know that the gain of the controller can be amplified up to a factor equal to K_c before the loop becomes unstable. K_c is given by*

$$
K_c = 10^{\frac{M_g}{20}} \tag{30}
$$

For this particular case, $K_c = 3.1623$. Hence, doubling the controller gain will not make the loop unstable. However, it is important to realize that the phase margin will normally diminish if we increase the gain.
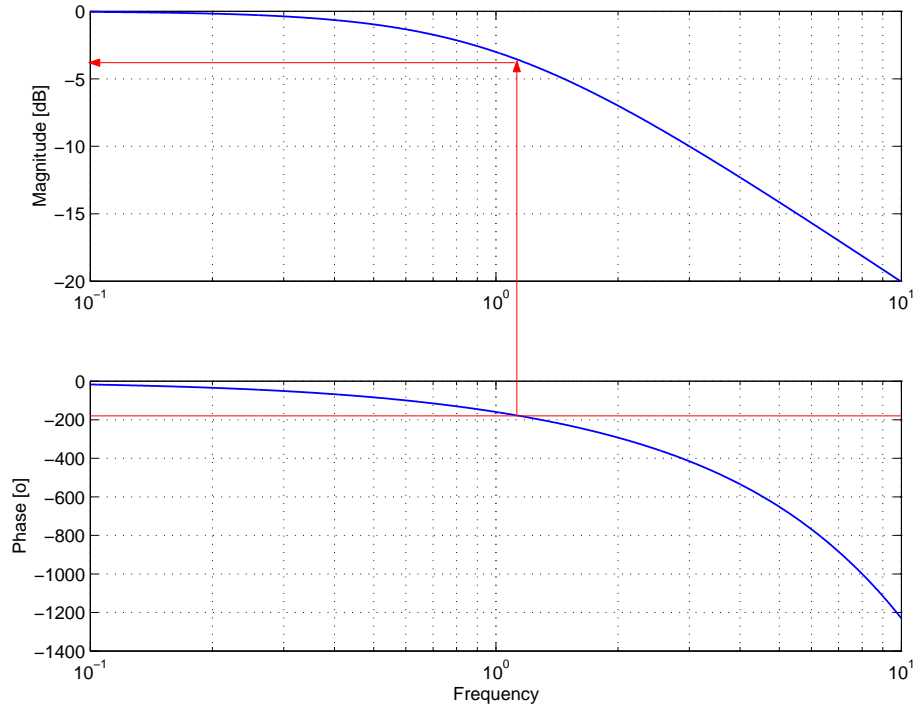

Figure 4: Bode diagrams for $F(s)$

Solution 5.8. *We first compute the open loop transfer function, $L_o(s) = G_o(s)C(s)$, which is given by*

$$L_o(s) = K\frac{s+b}{s(s+5)(s+a)} \tag{31}$$

5.8.1 *For $T_o(s)$ to be a canonical second order system we require $L_o(s)$ to have the form*

$$L_o(s) = \frac{\alpha}{s(s+\beta)} \implies T_o(s) = \frac{\alpha}{s^2+\beta s+\alpha} \tag{32}$$

We can then adjust α and β to satisfy the conditions specified regarding ω_n and ψ.

From the above equations we observe that if we choose $b = 5$, then $L_o(s)$ acquires the structure given in (32). With this choice

$$L_o(s) = \frac{K}{s(s+a)}; \quad and \; T_o(s) = \frac{K}{s^2+as+K} \tag{33}$$

If we now compare $T_o(s)$ with the transfer function in (4.8.4) of the book, we have that

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$$K = \omega_n^2 = 36 \tag{34}$$

$$a = 2\psi\omega_n = 7.2 \tag{35}$$

5.8.2 *We notice that the two inputs to the loop settle to constant values (the disturbance will converge to -3). Then all signals in the loop should converge to constant values, unless the loop is unstable. If the loop is unstable, signals in the loop will not settle and the concept of steady state becomes meaningless. In this example, we see that $T_o(s)$ is stable by design, and although there is a cancellation, it is a stable cancellation. Hence the loop is internally stable.*

One way to compute the steady state values is to calculate every transfer function and evaluate each of them at frequency zero. A different approach is followed here, which can be applied to a wide range of problems.

Consider the block diagram in Figure 5 where the steady state signals have been denoted by the suffix ∞. Also the control and plant blocks have been characterized by their corresponding d.c. gain.
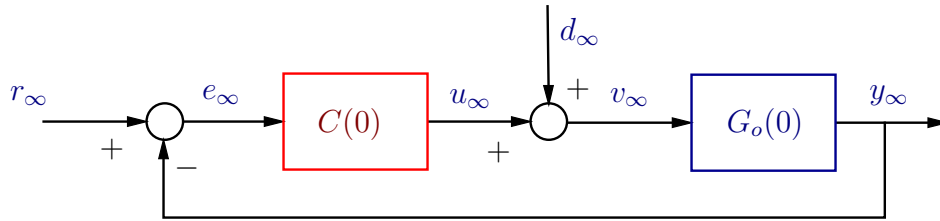

Figure 5: Feedback loop in steady state

We then proceed as follows:

1. *Since $G_o(0) = \infty$ and the loop is stable, then $v_\infty = 0$*

2. *$d_\infty = -3$ and $u_\infty = v_\infty - d_\infty = 3$*

3. *$C(0) = Kb/a = 25$ and $e_\infty = \dfrac{u_\infty}{C(0)} = 0.12$*

4. *$y_\infty = r_\infty - e_\infty = 4.88$*

We observe that, although there is integration in the loop (a pole at the origin), there is a non zero steady state error. This is due to the location of the pole at the origin and to the fact that we have an input disturbance. The reader might wish to repeat the computation assuming that the disturbance occurs at the plant output.

Alternatively, if the integration had been in the controller, the error would have converged to zero.

Solution 5.9. *The nominal sensitivity is given by*

$$S_o(s) = 1 - T_o(s) = \frac{s(s+30)}{s^2 + 30s + 400} \tag{36}$$

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5.9.1 *The (nominal) error is given by*

$$E_o(s) = S_o(s)R(s) \tag{37}$$

Therefore, in steady state, the error is a sinusoidal signal of the same frequency as the reference, but with different amplitude and phase. Using frequency response analysis we conclude that, in steady state

$$e_o(t) = |S_o(j4)|R_o \cos(4t + \angle S_o(j4)) \tag{38}$$

where $S_o(j\,4) = 0.3\angle 1.4$. Therefore

$$e_o(t) = 0.3R_o \cos(4t + 1.4) \tag{39}$$

The error is non zero since the sensitivity is not zero at $\omega = 4$.

5.9.2 *When the controller is applied to the calibration model, the error is given by*

$$E(s) = S(s)R(s) \tag{40}$$

On the other hand, using equations (5.9.15) and (5.9.19), we have that

$$S(s) = S_o(s)\frac{1}{1 + T_o(s)G_\Delta(s)} \tag{41}$$

Before we compute the steady state error signal, we need to check that the loop is robustly stable. We use Theorem 5.3 on page 146 of the book, replacing $|G_\Delta|$ by its upper bound. Thus, we need to prove that $|T_o(j\omega)\rho(j\omega)| < 1 \ \forall\omega$, since then we will then satisfy the sufficient condition for robust stability.

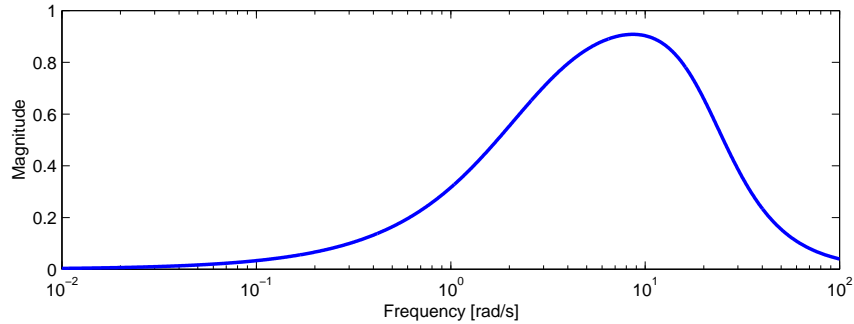*Figure 6 shows $|T_o(j\omega)\rho(j\omega)|$. Hence the system is stable.*

Figure 6: Frequency response of $T_o(j\omega)\rho(\omega)$

Therefore, the worst case will happen when $|G_\Delta(j\omega)| = \rho(\omega)$ and the phase of $T_o(j\,4)G_\Delta(j\,4)$ is an odd multiple of π. This leads to a sinusoidal steady state error where the amplitude $\hat{E}$ satisfies

$$\hat{E} \le |S_o(j\,4)|\frac{1}{1 - |T_o(j\,4)\rho(4)|}R_o = 0.3\frac{1}{1 - 0.795}R_o = 1.46R_o \tag{42}$$

We now observe that the error can be very large, and is much larger than the nominal error. This is due to the fact that the modeling error is significant at $\omega = 4$

Solution 5.10. *To answer the three questions above, we need to draw the Nyquist plot for $L(j\omega)$. This is done using the following MATLAB code[1]*

```
≫ L=tf([-0.5 0.5],[1 0.4 4]); w=logspace(-1,1,1000);
≫ Hl=freqresp(L,w);H=Hl(1,:);
≫ plot(H);hold;plot(conj(H));
```

5.10.1 *Using the MATLAB code above (and some other plotting commands to be used later), we obtain the plot shown in Figure 7.*
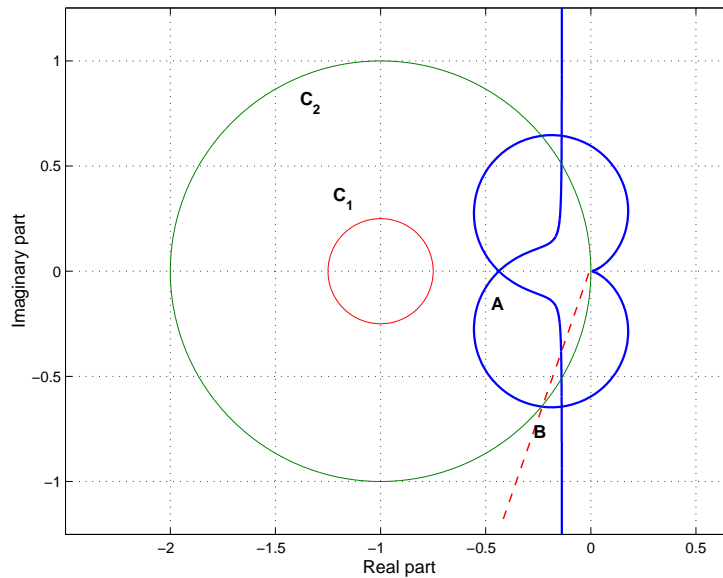

Figure 7: Nyquist plot of $L(j\omega)$

We know that the open loop transfer function $L(s)$ has no poles inside the modified Nyquist path (see Figure 5.6 of the book). From Figure 7 we also have that the polar plot does not encircle the origin. Thus the system is stable.

5.10.2 *The stability margins can be computed from Figure 7. The gain margin is determined from the crossing of the negative real axis. This point has been labeled **A**, which is approximately **(-0.43,0)**. Then*

$$M_g = -20\log(0.43) = 7.33[dB] \tag{43}$$

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[1]The MATLAB command **nyquist** is not suitable for functions with poles on the imaginary axis.

*The phase margin is determined by the intersection of the polar plot with the circle with unit radius and center at (-1,0). This circle has been labeled C_2 and the intersection point as **B**. Point **B** is approximately located at $(-0.23, -0.65)$. From this estimation and the definition in Figure 5.7 a) of the book, the phase margin is given by*

$$M_f = \arctan(0.65/0.23) = 1.23[rad] \tag{44}$$

5.10.3 *The sensitivity peak is given by the reciprocal of the distance from $(-1,0)$ to the polar plot. In Figure 7 we have drawn a circle, C_1, with radius 0.25 and center $(-1,0)$. Since the polar plot does not intersect this circle, then the sensitivity peak is smaller than 4. See Figure 5.7 b)in the book.*

Solution 5.11.

5.11.1 *The expression for the error is given by*

$$E(s) = S_o(s)R(s) + S_{io}(s)D_i(s) = S_o(s)R(s) + S_o(s)G_o(s)D_i(s) \tag{45}$$

Hence to have zero error when tracking the reference we require that $S_o(0) = 0$.

On the other hand, to fully compensate (in steady state) the disturbance, we require that $S_{io}(\pm j\omega_o) = 0$. Since the plant is stable and minimum phase, this condition translates into $S_o(\pm j\omega_o) = 0$.

Hence

$$G_o(0)C(0) = \infty \tag{46}$$
$$G_o(\pm j\omega_o)C(\pm j\omega_o) = 0 \tag{47}$$

Therefore, we require that the controller transfer function $C(s)$ has a pole at $s = 0$ and others at $s = \pm j\omega_o$. In other words, we will have a perfect inverse at these frequencies.

5.11.2 *If the nominal model is inaccurate, and assuming robust stability, we then have to check the calibration sensitivities (see equations (5.9.15) to (5.9.19) in the book). We recall that*

$$S(s) = \frac{S_o(s)}{1 + T_o(s)G_\Delta(s)} \tag{48}$$

From this equation we have that if $S_o(s_o) = 0$ for a given s_o, then $S(s_o) = 0$. Hence the error will still vanish asymptotically.