This is the support document for the proofs of Lemmas and Theorems in Paper "Optimal Design Of Linear Space Codes For Indoor MIMO Visible Light Communications With ML Detection" submitted to IEEE Photonics Journal

A. *Main Lemmas*

In order to prove Theorems 1 and 2, we need to establish the following six lemmas for the investigation of some important properties on the objective function and its feasible domain.

Lemma 1: Let θ_L and θ_U be defined by (8), where $0 < \lambda_2 \le \lambda_1$ and $0 \le \psi \le \frac{\pi}{2}$. Then, we have

$$0 \le \theta_U - \theta_L \le \frac{\pi}{2}$$

Moreover, $\theta_U - \theta_L \ge \frac{\pi}{3}$ if and only if the condition number of the channel satisfies the following inequality:

$$1 \le \frac{\lambda_1}{\lambda_2} \le \frac{1+\sqrt{1+3\sin(2\psi)}}{\sqrt{3}\sin(2\psi)}.$$

■

Proof: $\tan(\theta_U - \theta_L) = \frac{\tan(\theta_U)-\tan(\theta_L)}{1+\tan(\theta_U)\tan(\theta_L)} = \frac{2\lambda_1\lambda_2}{(\lambda_1^2-\lambda_2^2)\sin(2\psi)} > 0$, $\theta_L \in [-\pi/2, 0], \theta_U \in [0, \pi/2]$ and $0 \le \psi \le \frac{\pi}{2}$, we attain $0 \le \theta_U - \theta_L \le \frac{\pi}{2}$. Hence, requiring $\frac{\pi}{3} \le \theta_U - \theta_L \le \frac{\pi}{2}$ is equivalent to requiring $\tan(\theta_U - \theta_L) = \frac{2\lambda_1\lambda_2}{(\lambda_1^2-\lambda_2^2)\sin(2\psi)} \ge \sqrt{3}$, which, in turn, is also equivalent to the following inequality: $(\sqrt{3}\sin(2\psi))\rho^2 - 2\rho - \sqrt{3}\sin(2\psi) \le 0$, where $\rho = \frac{\lambda_1}{\lambda_2}$ is the condition number of the channel $\mathbf{H}$. Therefore, we have $1 \le \frac{\lambda_1}{\lambda_2} \le \frac{1+\sqrt{1+3\sin(2\psi)}}{\sqrt{3}\sin(2\psi)}$. It is not difficult to examine $\frac{1+\sqrt{1+3\sin(2\psi)}}{\sqrt{3}\sin(2\psi)} \ge 1$. This completes the proof of Lemma 1. □

In (α, β)-plane, whether or not there is any intersection between Lines $|\alpha - \beta| = \frac{\pi}{3}$ and the square Ω as shown in Fig. 4 plays a crucial role in obtaining a closed-from solution to Problem 2. Lemma 1 provides us with a necessary and sufficient condition to check when there exists an intersection. The following Lemmas 2- 6 will lead us how to find the optimal solution under this condition.

Lemma 2: For $\psi \in [0, \frac{\pi}{2}]$, let three angles θ, ϕ and τ be defined, respectively, by

$$\theta = \arctan\left(\frac{\lambda_1}{\lambda_2} \times \tan\left(\frac{\pi}{4} - \psi\right)\right) \quad (14)$$

$$\phi = \arctan\left(\frac{\sin 2(\theta_L - \theta)}{2 + \cos 2(\theta_L - \theta)}\right) \quad (15)$$

$$\tau = \arctan\left(\frac{\sin 2(\theta_U - \theta)}{2 + \cos 2(\theta_U - \theta)}\right), \quad (16)$$

where θ_L and θ_U are given in (8). Then, we have the following two statements:

1) When $0 \le \psi < \frac{\pi}{4}$, we have $\theta_U - \frac{\pi}{3} \le \theta$ and $\phi \ge -\frac{\pi}{6}$ if $\theta - \theta_L \ge \frac{\pi}{3}$.
2) When $\frac{\pi}{4} \le \psi \le \frac{\pi}{2}$, we have $\theta_L + \frac{\pi}{3} \ge \theta$ and $\tau \le \frac{\pi}{6}$ if $\theta_U - \theta \ge \frac{\pi}{3}$. ■

Proof: The whole proof captures the following two parts.

Proof of Statement 1: Since

$$\tan(\theta_U - \theta) = \frac{\tan\theta_U - \tan\theta}{1 + \tan\beta\tan\theta} = \frac{\frac{\lambda_2}{\lambda_1}\cot\psi - \frac{\lambda_1}{\lambda_2}\tan\left(\frac{\pi}{4} - \psi\right)}{1 + \cot\psi\tan\left(\frac{\pi}{4} - \psi\right)} \quad (17)$$

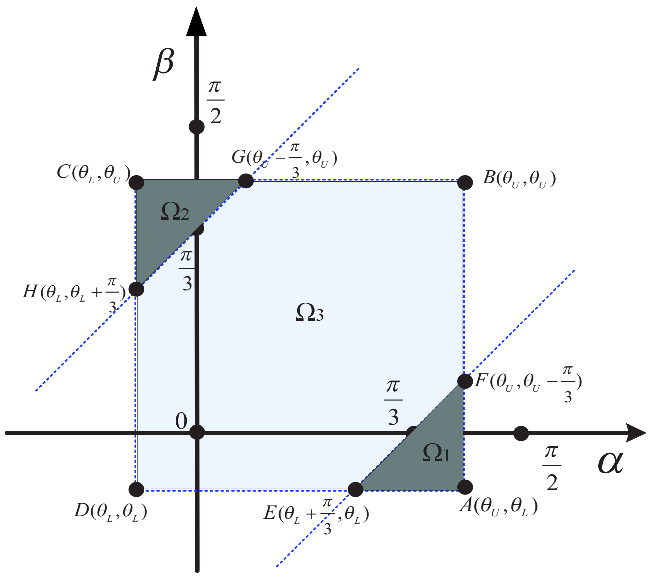

Fig. 4. Feasible domain $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ in terms of α and β has intersections with Lines $|\beta - \alpha| = \frac{\pi}{3}$.

and $\theta_U - \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, in this case, $\theta_U - \frac{\pi}{3} \le \theta$ is equivalent to $\theta_U - \theta < \frac{\pi}{3}$, which, together with (17), is equivalent to saying

$$\rho^2\tan\left(\frac{\pi}{4} - \psi\right) + \sqrt{3}\Big(1 + \cot\psi\tan\left(\frac{\pi}{4} - \psi\right)\Big)\rho - \cot\psi \ge 0, \quad (18)$$

where $\rho = \frac{\lambda_1}{\lambda_2}$. Hence, we have

$$\rho \ge \max\left\{1, \frac{-\sqrt{3}\Big(1 + \cot\psi\tan\left(\frac{\pi}{4} - \psi\right)\Big) + \sqrt{\mathcal{D}_U}}{2\tan\left(\frac{\pi}{4} - \psi\right)}\right\}, \quad (19)$$

where the discriminant is $\mathcal{D}_U = 3(1 + \cot\psi\tan\left(\frac{\pi}{4} - \psi\right))^2 + 4\cot\psi\tan\left(\frac{\pi}{4} - \psi\right) > 0$. Now, we need to prove that

$$\frac{-\sqrt{3}\Big(1 + \cot\psi\tan\left(\frac{\pi}{4} - \psi\right)\Big) + \sqrt{\mathcal{D}_U}}{2\tan\left(\frac{\pi}{4} - \psi\right)} < 1. \quad (20)$$

To do that, we notice that

$$\begin{aligned} &\frac{-\sqrt{3}\Big(1 + \cot\psi\tan\left(\frac{\pi}{4} - \psi\right)\Big) + \sqrt{\mathcal{D}_U}}{2\tan\left(\frac{\pi}{4} - \psi\right)} \\ &= \frac{2\cot\psi}{\sqrt{3}\Big(1 + \cot\psi\tan\left(\frac{\pi}{4} - \psi\right)\Big) + \sqrt{\mathcal{D}_U}} \\ &\le \frac{\cot\psi}{\sqrt{3}\Big(1 + \cot\psi\tan\left(\frac{\pi}{4} - \psi\right)\Big)}, \end{aligned} \quad (21a)$$

since $\mathcal{D}_U \ge \sqrt{3}\Big(1 + \cot\psi\tan\left(\frac{\pi}{4} - \psi\right)\Big)$. In addition, we further note that

$$\begin{aligned} &\frac{\cot\psi}{\sqrt{3}\Big(1 + \cot\psi\tan\left(\frac{\pi}{4} - \psi\right)\Big)} = \frac{1}{\sqrt{3}} \times \frac{\cot^2\psi + \cot\psi}{\cot^2\psi + 1} \\ &= \frac{1}{\sqrt{3}} \times \left(1 + \frac{\cot\psi - 1}{\cot^2\psi + 1}\right) < \frac{1}{\sqrt{3}} \times \left(1 + \frac{\cot\psi}{\cot^2\psi + 1}\right) \\ &= \frac{1}{\sqrt{3}} \times \left(1 + \frac{1}{\cot\psi + \cot^{-1}\psi}\right) \le \frac{\sqrt{3}}{2} < 1. \end{aligned} \quad (21b)$$

Now, combining (20) and (21), we claim that (20) is indeed true. This, together with (19) gives us $\rho \geq 1$, which implies $\theta_U - \frac{\pi}{3} \leq \theta$.

In addition, on one hand, we note that $\theta - \theta_L \in [0, \pi]$ and $\cot(x)$ is monotonically decreasing in $[0, \pi]$. On the other hand, we also notice that $\tan(\phi) = \frac{\sin 2(\theta_L - \theta)}{2 + \cos 2(\theta_L - \theta)} = -\frac{2t}{3t^2+1} = g_L(t)$, where $t = \cot(\theta - \theta_L)$, and that $g_L'(t) = \frac{3t^2-1}{(3t^2+1)^2}$. Hence, $g_L(t)$ is monotonically increasing when $|t| > \frac{\sqrt{3}}{3}$, which is equivalent to $\frac{2\pi}{3} < \theta - \theta_L < \pi$, and monotonically decreasing when $|t| < \frac{\sqrt{3}}{3}$, i.e., $\frac{\pi}{3} < \theta - \theta_L < \frac{2\pi}{3}$. However, at any rate, we always have $\tan(\phi) \geq g_L\left(\frac{\sqrt{3}}{3}\right) = -\frac{\sqrt{3}}{3}$. This, along with $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ infers $\phi \geq -\frac{\pi}{6}$. So far, we have completed the proof of Statement 1.

Proof of Statement 2: When $\frac{\pi}{4} \leq \psi \leq \frac{\pi}{2}$, $\frac{\pi}{2} \leq \theta \leq 0$ and as a result, $\theta_L - \theta, -\frac{\pi}{3} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore, inequality $\theta_L + \frac{\pi}{3} \geq \theta$ is equivalent to $\theta_L - \theta \geq -\frac{\pi}{3}$, which, in turn, is also equivalent to $\tan(\theta_L - \theta) \geq -\sqrt{3}$. Since $\tan(\theta_L - \theta) = \frac{\tan\theta_L - \tan\theta}{1 + \tan\theta_L \tan\theta} = \frac{-\frac{\lambda_2}{\lambda_1}\tan\psi - \frac{\lambda_1}{\lambda_2}\tan\left(\frac{\pi}{4} - \psi\right)}{1 - \tan\psi \tan\left(\frac{\pi}{4} - \psi\right)}$, $\theta_L - \theta \geq -\frac{\pi}{3}$ if and only if

$$\rho^2 \tan\left(\frac{\pi}{4} - \psi\right) - \sqrt{3}\Big(1 - \tan\psi \tan\left(\frac{\pi}{4} - \psi\right)\Big)\rho + \tan\psi \leq 0, \quad (22)$$

where $\rho = \frac{\lambda_1}{\lambda_2}$. Now, let $\psi = \frac{\pi}{2} - \tilde{\psi}$. Then, $0 \leq \tilde{\psi} \leq \frac{\pi}{4}$ and (22) becomes the exact same as (18) except for the fact that ψ is replaced by $\tilde{\psi}$. Hence, following the same way, we can prove $\theta_L + \frac{\pi}{3} \geq \theta$.

Furthermore, we notice that $\theta_U - \theta \in [0, \pi]$ and $\cot(x)$ is monotonically decreasing in $[0, \pi]$. On the other hand, we also notice that $\tan(\tau) = \frac{\sin 2(\theta_U - \theta)}{2 + \cos 2(\theta_U - \theta)} = \frac{2t}{3t^2+1} = g_U(t)$, where $t = \cot(\theta_U - \theta)$, and that $g_U'(t) = -\frac{3t^2-1}{(3t^2+1)^2}$. Hence, $g_U(t)$ is monotonically decreasing when $|t| > \frac{\sqrt{3}}{3}$, which is equivalent to $\frac{2\pi}{3} < \theta - \theta_L < \pi$, and monotonically increasing when $|t| < \frac{\sqrt{3}}{3}$, i.e., $\frac{\pi}{3} < \theta - \theta_L < \frac{2\pi}{3}$. However, in either case, we can always obtain $\tan(\phi) \leq g_U\left(\frac{\sqrt{3}}{3}\right) = \frac{\sqrt{3}}{3}$. This, along with $\tau \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ implies $\tau \leq \frac{\pi}{6}$. Thus far, we have completed the proof of Statement 2 and thus, Lemma 2. □

Lemma 3: Let function $F_1(\alpha, \beta)$ be defined by (11a). Then, the minimum of $F_1(\alpha, \beta)$ in $\Omega_1 \cup \Omega_2$ as shown in Fig. 1 is achieved at one of the six vertices, i.e., $\min_{(\alpha,\beta)\in\Omega_1\cup\Omega_2} F_1(\alpha, \beta) = \min\{F_1(C), F_1(G), F_1(H), F_1(E), F_1(F), F_1(A)\}$. ■

Proof: Taking partial derivatives on both sides of (12a), we have $\frac{\partial f_1}{\partial \alpha} = -\sin(\alpha - \theta) = 0$ and $\frac{\partial f_1}{\partial \beta} = -\sin(\beta - \theta) = 0$. Hence, $\alpha - \theta = k\pi$ and $\beta - \theta = \ell\pi$, where both of k and ℓ are integers. Since $\alpha - \theta \in [-\pi/2, \pi/2], \beta - \theta \in [-\pi/2, \pi/2]$, and $|\alpha - \beta| \leq \theta_U - \theta_L \leq \frac{\pi}{2}$, we have $k = \ell = 0$. However, at this point, $f_1(\alpha, \beta)$ achieves its maximum value. Since $f_1(\alpha, \beta)$ is continuous in the compact domain $\Omega_1 \cup \Omega_2$, it must have the minimum value, which is achieved on the boundary of $\Omega_1 \cup \Omega_2$. In addition, since $f_1(\alpha, \beta)$ is symmetrical in its feasible domain $\Omega_1 \cup \Omega_2$ with respect to the line of $\alpha = \beta$, we only need consider one of triangular domains, say Ω_2 as shown in Figure 4. Let us now consider the following three lines.

1) Line HC. On this line, the objective function $f_1(\alpha, \beta)$ is reduced to $f_1(\theta_L, \beta) = \cos(\theta_L - \theta) + \cos(\beta - \theta)$, where $\theta_L + \frac{\pi}{3} \leq \beta \leq \theta_U$. Since $\beta - \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\cos(\beta - \theta)$ is monotonically increasing for $\beta < \theta$, and is monotonically decreasing for $\beta > \theta$, in this case, $\min f_1(\alpha, \beta) = \min\{f_1(H), f_1(C)\}$.

2) Line CG. On this line, the objective function $f_1(\alpha, \beta)$ becomes $f_1(\theta_L, \beta) = \cos(\alpha - \theta) + \cos(\theta_U - \theta)$, where $\theta_L \leq \alpha \leq \theta_U - \frac{\pi}{3}$. In the same token, Since $\cos(\alpha - \theta)$ is monotonically increasing for $\alpha < \theta$, and is monotonically decreasing for $\alpha > \theta$, in this case, $\min f_1(\alpha, \beta) = \min\{f_1(C), f_1(G)\}$.

3) Line HG. On this line, the objective function $f_1(\alpha, \beta)$ is simplified into $f_1(\alpha, \beta) = \cos(\theta_L - \theta) + \cos(\beta - \theta) = \sqrt{3}\cos\left(\beta - \frac{\pi}{6} - \theta\right)$, where $\theta_L + \frac{\pi}{3} \leq \beta \leq \theta_U$. Similarly, since $\cos(\beta - \frac{\pi}{6} - \theta)$ is monotonically increasing for $\beta < \theta + \pi/6$, and is monotonically decreasing for $\beta > \theta + \pi/6$, in this situation, $\min f_1(\alpha, \beta) = \min\{f_1(H), f_1(G)\}$.

Summarizing the above discussions, we conclude that $f_1(\alpha, \beta)$ achieves its minimum at one of the three vertices of the triangular Ω_2, i.e., $\min f_1(\alpha, \beta) = \min\{f_1(H), f_1(C), f_1(G)\}$. This completes the proof of Lemma 3. □

Lemma 4: Let function $F_2(\alpha, \beta)$ be defined by (11b). Then, the minimum of $F_2(\alpha, \beta)$ in Ω_3 as shown in Fig. 1 is achieved at one of the six vertices, i.e., $\min_{(\alpha,\beta)\in\Omega_3} F_2(\alpha, \beta) = \min\{F_2(B), F_2(D), F_2(G), F_2(H), F_2(E), F_2(F)\}$. ■

Proof: Taking partial derivatives on both sides of (12b) yield $\frac{\partial f_2}{\partial \alpha} = -2\sin(2\alpha - \beta - \theta) = 0$ and $\frac{\partial f_2}{\partial \beta} = -\sin(2\alpha - \beta - \theta) - \sin(\beta - \theta) = 0$, which gives us $\sin(2\alpha - \beta - \theta) = \sin(\beta - \theta) = 0$. Hence, $2\alpha - \beta - \theta = k\pi$ and $\beta - \theta = \ell\pi$, where both of k and ℓ are integers. Since $\alpha - \theta \in [-\pi/2, \pi/2], \beta - \theta \in [-\pi/2, \pi/2]$, and $|\alpha - \beta| \leq \theta_U - \theta_L \leq \frac{\pi}{2}$, we have $k = \ell = 0$ and thus, $\alpha = \beta = \theta$. At this point, $f_2(\alpha, \beta)$ achieves its maximum value. Since $f_2(\alpha, \beta)$ is continuous in the compact domain Ω_3, it must have the minimum value, which is achieved on the boundary of Ω_3. The boundary consists of six lines: GB, BF, FE, ED, DH and HG as shown in Figure 4. Notice that on Lines FE and HG, $f_1(\alpha, \beta) = f_2(\alpha, \beta)$, which has been discussed in Lemma 3. In addition, on Line GB, $f_2(\alpha, \theta_U) = \cos(2\alpha - \theta_U - \theta) + 2\cos(\theta_U - \theta)$ and on Line ED, $f_2(\alpha, \theta_U) = \cos(2\alpha - \theta_U - \theta) + 2\cos(\theta_U - \theta)$, which can also be dealt with by following the way similar to the proof of Lemma 3. Hence, in the following we only need to consider the other two lines, i.e., BF and DH.

1) Line BF. On this line, the objective function $f_2(\alpha, \beta)$ is reduced to $f_2(\theta_U, \beta) = \cos(2\theta_U - \beta - \theta) + \cos(\beta - \theta) = \Big(2 + \cos 2(\theta_U - \theta)\Big)\cos(\beta - \theta) + \sin 2(\theta_U - \theta)\sin(\beta - \theta) = \sqrt{\Big(2 + \cos 2(\theta_U - \theta)\Big)^2 + \sin^2 2(\theta_U - \theta)}\cos(\beta - \tau - \theta)$, where $\theta_U - \frac{\pi}{3} \leq \beta \leq \theta_U$ and $\tau = \arctan\left(\frac{\sin 2(\theta_U - \theta)}{2 + \cos 2(\theta_U - \theta)}\right)$. Now, following the same argument as in the proof of Lemma 3, we can conclude that in this situation, $\min f_2(\alpha, \beta) = \min\{f_2(B), f_2(F)\}$.

2) Line DH. On this line, the objective function $f_2(\alpha, \beta)$ becomes $f_2(\theta_L, \beta) = \cos(2\theta_L - \beta - \theta) + \cos(\beta - \theta) = \sqrt{\Big(2 + \cos 2(\theta_L - \theta)\Big)^2 + \sin^2 2(\theta_L - \theta)}\cos(\beta - \phi - \theta)$, where $\theta_L \leq \beta \leq \theta_L + \frac{\pi}{3}$ and $\phi = \arctan\left(\frac{\sin 2(\theta_L - \theta)}{2 + \cos 2(\theta_L - \theta)}\right)$. Following the discussion similar to Line BF, we can claim

that in this case, $\min f_2(\alpha,\beta) = \min\{f_2(D), f_2(H)\}$.

Summarizing the above discussions, we conclude that $f_2(\alpha,\beta)$ achieves its minimum at one of the six vertices of Ω_2, i.e., $\min f_2(\alpha,\beta) = \min\{f_2(G), f_2(B), f_2(F), f_2(E), f_2(D), f_2(H)\}$. This completes the proof of Lemma 4. $\square$

Lemma 5: Let a, b and c be given three positive real numbers with $\frac{1}{2} \le c \le 1$. If a function $f(x)$ is defined on a closed interval $[1, 2c]$ by $f(x) = \frac{x^2-2cx+1}{(ax+b)^2}$, then, $f(x)$ achieves its maximum at $x = 2c$, its minimum at $x = 1$ when $b \ge a$ and at $x = x_0$ when $b \le a$, where $x_0 = \frac{a+bc}{b+ac}$. $\blacksquare$

Proof: It is not difficult to compute the first order derivative with respect to x such that $f'(x) = \frac{2(b+ac)\left(x-\frac{a+bc}{b+ac}\right)}{(ax+b)^3}$. Now, it can be seen that there is only one possible root x_0 for $f'(x) = 0$. We need to know when x_0 belongs to an open interval $(1, 2c)$. Hence, this leads us to considering the following two situations:

1) $b > a$. In this case, it can be observed that $x_0 < 1$ and thus, $f'(x) > 0$, i.e., $f(x)$ is monotonically increasing. Therefore, $\min f(x) = f(1)$ and $\max f(x) = f(2c)$.

2) $b \le a$. In this case, we first know that $x_0 \ge 1$. On the other hand, judging whether $x_0 \le 2c$ is equivalent to examining whether the following inequality holds:

$$\frac{a+bc}{b+ac} \le 2c, \tag{23}$$

which is $2ac^2 + bc - a \ge 0$. There are two roots for $2ax^2 + bx - a = 0$ in terms of x, i.e., $x_1 = \frac{-b-\sqrt{b^2+8a^2}}{4a}$ and $x_2 = \frac{-b+\sqrt{b^2+8a^2}}{4a}$. Since $x_2 < \frac{1}{4}$ and $c \ge \frac{1}{2}$, c indeed meets (23). Therefore, $f(x)$ achieves its minimum at $x = x_0$ and its maximum at one of the two ending points: $x = 1$ and $x = 2c$. Notice that $f(1) = \frac{2(1-c)}{(a+b)^2}$ and $f(2c) = \frac{1}{(2ac+b)^2}$. Now, we need to prove that $f(1) \le f(2c)$, which is equivalent to proving that the following inequality is true:

$$\frac{2(1-c)}{(a+b)^2} \le \frac{1}{(2ac+b)^2} \tag{24}$$

This leads us to considering a function $\tilde{f}(x) = 2(1-x)(2ax+b)^2$ for $1/2 \le x \le 1$. Since $\tilde{f}'(x) = -2(2ax+b)(4ax-(2a-b))$, $2ax+b > 0$ and $4ax-(2a-b) \ge 2a-(2a-b) = b > 0$, $\tilde{f}'(x) < 0$ and hence, $\tilde{f}(x)$ is monotonically decreasing. Since $\tilde{f}(1/2) = (a+b)^2$, $\tilde{f}(c) \le \tilde{f}(1) = (a+b)^2$, implying that c indeed satisfies (24). This completes the proof of Lemma 5. $\square$

Lemma 6: For $0 \le \psi \le \frac{\pi}{2}$, let functions $F_1(\alpha,\beta)$ and $F_2(\alpha,\beta)$ be defined by (11). Then, we have

$$\begin{aligned}
F_1(C) &= \frac{1}{\sqrt{\lambda_1^2\cos^2\psi+\lambda_2^2\sin^2\psi}} + \frac{1}{\sqrt{\lambda_1^2\sin^2\psi+\lambda_2^2\cos^2\psi}}\\
F_2(A) &= \frac{2(\lambda_1^2-\lambda_2^2)\sin 2\psi}{\sqrt{\lambda_1^2\sin^2\psi+\lambda_2^2\cos^2\psi}\sqrt{(\lambda_1^2-\lambda_2^2)^2\sin^2 2\psi+4\lambda_1^2\lambda_2^2}}\\
&\quad + \frac{1}{\sqrt{\lambda_1^2\cos^2\psi+\lambda_2^2\sin^2\psi}}\\
F_2(B) &= \frac{3}{\sqrt{\lambda_1^2\sin^2\psi+\lambda_2^2\cos^2\psi}}\\
F_2(C) &= \frac{2(\lambda_1^2-\lambda_2^2)\sin 2\psi}{\sqrt{\lambda_1^2\cos^2\psi+\lambda_2^2\sin^2\psi}\sqrt{(\lambda_1^2-\lambda_2^2)^2\sin^2 2\psi+4\lambda_1^2\lambda_2^2}}\\
&\quad + \frac{1}{\sqrt{\lambda_1^2\sin^2\psi+\lambda_2^2\cos^2\psi}}\\
F_2(D) &= \frac{3}{\sqrt{\lambda_1^2\cos^2\psi+\lambda_2^2\sin^2\psi}}
\end{aligned}$$

Moreover, the following four statements are true:

1) $F_2(B) \ge F_2(D)$ if and only if $0 \le \psi \le \frac{\pi}{4}$.
2) $F_1(C) \ge F_2(D)$ if and only if $0 \le \psi < \arctan\left(\frac{1}{2}\right)$ and
$$\frac{\lambda_1}{\lambda_2} \ge \sqrt{\frac{4-5\sin^2\psi}{1-5\sin^2\psi}}.$$
3) $F_1(C) \ge F_2(B)$ if and only if $\psi > \arctan 2$ and
$$\frac{\lambda_1}{\lambda_2} \ge \sqrt{\frac{5\sin^2\psi-1}{5\sin^2\psi-4}}.$$
4) Under the condition of
$$\frac{\lambda_1}{\lambda_2} > \frac{1+\sqrt{1+3\sin(2\psi)}}{\sqrt{3}\sin(2\psi)},$$
$F_2(A) \ge F_2(C)$ if and only if $0 \le \psi \le \frac{\pi}{4}$. $\blacksquare$

Proof: For presentation clarity, we discuss how to prove each statement separately.

Proof of Statement 1: $F_2(B) \ge F_2(D)$ if and only if $\sqrt{\lambda_1^2\sin^2\psi+\lambda_2^2\cos^2\psi} \le \sqrt{\lambda_1^2\cos^2\psi+\lambda_2^2\sin^2\psi}$. This is equivalent to $(\lambda_1^2-\lambda_2^2)\cos 2\psi \ge 0$, which is equivalent to $0 \le \psi \le \frac{\pi}{4}$, since $\lambda_1 > \lambda_2 > 0$ and $0 \le \psi \le \frac{\pi}{2}$.

Proof of Statement 2: $F_1(C) \ge F_2(D)$ if and only if $2\sqrt{\lambda_1^2\sin^2\psi+\lambda_2^2\cos^2\psi} \le \sqrt{\lambda_1^2\cos^2\psi+\lambda_2^2\sin^2\psi}$. This, in turn, is equivalent to $(5\sin^2\psi-1)\rho^2 \le 5\sin^2\psi-4$, where $\rho = \frac{\lambda_1}{\lambda_2} \ge 1$. Therefore, $5\sin^2\psi - 1 < 0$ and $\rho \ge \sqrt{\frac{4-5\sin^2\psi}{1-5\sin^2\psi}}$. This completes the proof of Statement 2.

Proof of Statement 3: It can be proved by following the way similar to the proof of Statement 2.

Proof of Statement 4: First, we note that

$$\begin{aligned}
F_2(A) - F_2(C) &= \frac{F_2(B)-F_2(D)}{3}\\
&\times\left(\frac{2(\lambda_1^2-\lambda_2^2)\sin 2\psi}{\sqrt{(\lambda_1^2-\lambda_2^2)^2\sin^2 2\psi+4\lambda_1^2\lambda_2^2}} - 1\right)
\end{aligned}\tag{25}$$

On the other hand, we also notice that inequality $\frac{2(\lambda_1^2-\lambda_2^2)\sin 2\psi}{\sqrt{(\lambda_1^2-\lambda_2^2)^2\sin^2 2\psi+4\lambda_1^2\lambda_2^2}} > 1$ is equivalent to $\sqrt{3}(\lambda_1^2 - \lambda_2^2)\sin 2\psi \ge 2\lambda_1\lambda_2$, which is $(\sqrt{3}\sin 2\psi)\rho^2 - 2\rho -$

$\sqrt{3}\sin 2\psi > 0$, where $\rho = \frac{\lambda_1}{\lambda_2}$. Hence, we have $\rho > \frac{1+\sqrt{1+3\sin(2\psi)}}{\sqrt{3}\sin(2\psi)}$. This is our assumption. Therefore, under this condition, equation (25) is equivalent to $F_2(B) \geq F_2(D)$, which is Statement 1. This completes the proof of Statement 4 and thus, Lemma 6. □

B. Proofs of Theorems

1) Proof of Theorem 1: Without loss of generality, we assume that $d_1 \geq d_2$ and thus, we only need to consider the following three cases:

$$\begin{aligned} &1)\quad d_3^2 \geqslant d_1^2 \geqslant d_2^2, \quad \max d_2 \\ &2)\quad d_1^2 \geqslant d_3^2 \geqslant d_2^2, \quad \max d_2 \\ &3)\quad d_1^2 \geqslant d_2^2 \geqslant d_3^2. \quad \max d_3 \end{aligned}$$

In addition, It is not difficult to observe that the optimal solution cannot be achieved when $d_2 = 0$. Hence, to facilitate our analysis, by utilizing the parameterization of the space coding matrix $\mathbf{F}$ proposed in Subsection III-B, Cases 1), 2) and 3) can be equivalently represented in terms of μ, α and βm respectively, as

$$\begin{aligned} &a)\quad 1 \leq \mu \leq \frac{1}{2\cos(\alpha-\beta)}, \quad \max d_2 \\ &b)\quad \mu \geq \max\left\{\frac{1}{2\cos(\alpha-\beta)}, 2\cos(\alpha-\beta)\right\}, \quad \max d_2 \\ &c)\quad 1 \leq \mu \leq 2\cos(\alpha-\beta), \quad \max d_3, \end{aligned}$$

where $(\alpha,\beta) \in \Omega$, i.e., $\theta_L \leq \alpha, \beta \leq \theta_U$. Now, it is more clear to see that in order to solve these optimization problems, we need to consider when there is an intersection between the square Ω and Lines: $|\alpha-\beta| = \frac{\pi}{3}$.

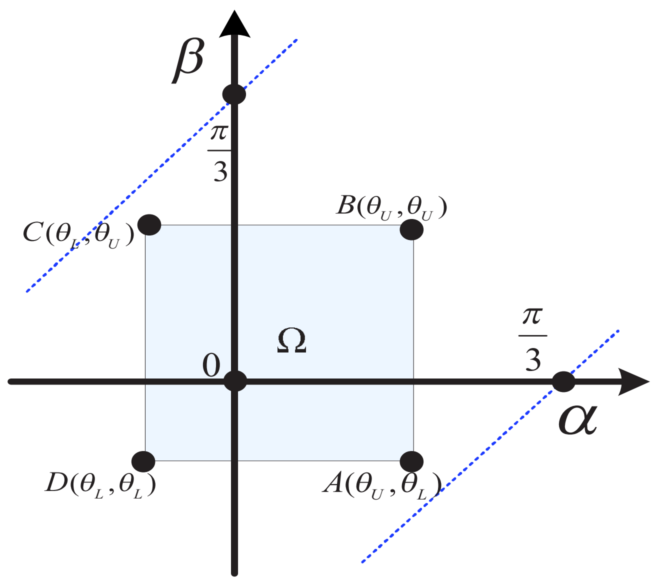

Fig. 5. Feasible domain Ω in terms of α and β has no intersection with Lines $|\beta-\alpha| = \frac{\pi}{3}$.

(I) **No intersection**. By Lemma 1, we know that there is no intersection, as shown in Fig. 5, between the square Ω and Lines: $|\alpha-\beta| = \frac{\pi}{3}$ if $0 \leq \theta_U - \theta_L < \frac{\pi}{3}$, which is equivalent to the fact that the condition number of the channel satisfies the following inequality: $\frac{\lambda_1}{\lambda_2} > \frac{1+\sqrt{1+3\sin(2\psi)}}{\sqrt{3}\sin(2\psi)}$. Under this condition, the feasible domain of Case a) is empty. Therefore, we only need to consider Cases b) and c), each feasible set of which is the whole square Ω.

Case b): $\mu \geq 2\cos(\alpha-\beta)$. In this case, we need to maximize d_2. Using (10), we have $\max d_2 = \max_{\alpha,\beta}\max_{\mu}\frac{2P_{\mathrm{T}}}{F(\alpha,\beta,\mu)} = \frac{2P_{\mathrm{T}}}{\min_{\alpha,\beta}\min_{\mu}F(\alpha,\beta,\mu)} = \frac{2P_{\mathrm{T}}}{\min_{\alpha,\beta}F_2(\alpha,\beta)}$, where we have utilized the fact that $\min_{\mu}F(\alpha,\beta,\mu) = F(\alpha,\beta,2\cos(\alpha-\beta))$, since $\mu \geq 2\cos(\alpha-\beta)$ and $F(\alpha,\beta,\mu)$ is an increasing function of μ. Now, following the discussion similar to the proof of Lemma 4, we can conclude that $F_2(\alpha,\beta)$ achieves its minimum at one of the four vertices of Ω, i.e., $\min F_2(\alpha,\beta) = \min\{F_2(A), F_2(B), F_2(C), F_2(D)\}$.

Case c): $1 \leq \mu \leq 2\cos(\alpha-\beta)$. In this case, we need to maximize d_3. Combining the definition of d_3 and (10), we can represent d_3^2 as $d_3^2 = d_2^2(\mu^2 - 2\mu\cos(\alpha-\beta)+1) = \frac{4P_{\mathrm{T}}^2\left(\mu^2-2\mu\cos(\alpha-\beta)+1\right)}{F^2(\alpha,\beta,\mu)} = \frac{4P_{\mathrm{T}}^2}{\Delta^2(\lambda_1,\lambda_2,\psi)}\frac{\mu^2-2\mu\cos(\alpha-\beta)+1}{f^2(\alpha,\beta,\mu)}$. Hence, we have $\max d_3^2 = \max_{\alpha,\beta}\max_{\mu} d_3^2$. Now, applying Lemma 5 into function $\frac{\mu^2-2\mu\cos(\alpha-\beta)+1}{f^2(\alpha,\beta,\mu)}$ yields $\max_{\mu} d_3^2 = \frac{4P_{\mathrm{T}}^2}{\Delta^2(\lambda_1,\lambda_2,\psi)}\frac{1}{f^2(\alpha,\beta,2\cos(\alpha,\beta))} = \frac{4P_{\mathrm{T}}^2}{F^2(\alpha,\beta,2\cos(\alpha,\beta))} = \frac{4P_{\mathrm{T}}^2}{F_2^2(\alpha,\beta)}$. Therefore, $\max d_3 = \frac{2P_{\mathrm{T}}}{\min F_2(\alpha,\beta)}$, which is exactly the same case as Case b).

Now, summarizing the above discussions on Cases b) and c), we attain that the optimum is reached when $d_2 = d_3$ and is given by $\frac{2P_{\mathrm{T}}}{\min F_2(\alpha,\beta)}$, where $\min F_2(\alpha,\beta) = \min\{F_2(A), F_2(B), F_2(C), F_2(D)\}$. Combining this with Lemma 6, we have $\min F_2(\alpha,\beta) = \min\{F_2(C), F_2(D)\}$ when $0 \leq \psi \leq \frac{\pi}{4}$. On the other hand, we know that in this case, $f_2(\theta_L,\beta) = \sqrt{\left(2+\cos 2(\theta_L-\theta)\right)^2 + \sin^2 2(\theta_L-\theta)\cos(\beta-\phi-\theta)}$, where $\theta_L \leq \beta \leq \theta_U$. From the definitions of θ and ϕ in Lemma 2, when $0 \leq \psi \leq \frac{\pi}{4}$, we can obtain $\tan\phi = \frac{2\tan(\theta_L-\theta)}{3+2\tan^2(\theta_L-\theta)} \geq \tan(\theta_L-\theta)$ and as a consequence, $\theta_L \leq \theta_L + \phi$. Similarly, we can prove $\theta_U \geq \theta + \phi$. Hence, when $0 \leq \psi \leq \frac{\pi}{4}$ and $\theta + \phi - \theta_L \geq \theta_U - (\theta+\phi)$, i.e., $\theta + \phi \geq \frac{\theta_L+\theta_U}{2}$, we have $f_2(\theta_L,\theta_L) \leq f_2(\theta_L,\theta_U)$, i.e., $F_2(D) \leq F_2(C)$ and thus, $\min F_2(\alpha,\beta) = F_2(D)$. Analogously, we can prove the case where $\frac{\pi}{4} < \psi \leq \frac{\pi}{2}$.

(II) **Intersection**. By Lemma 1, there is an intersection, as shown in Fig. 4, between the square Ω and Lines: $|\alpha-\beta| = \frac{\pi}{3}$ if $\theta_U - \theta_L \geq \frac{\pi}{3}$, which is equivalent to the fact that the condition number of the channel satisfies $1 \leq \frac{\lambda_1}{\lambda_2} \leq \frac{1+\sqrt{1+3\sin(2\psi)}}{\sqrt{3}\sin(2\psi)}$. Under this condition, different cases have different feasible sets. So we need to deal with each individual optimization problem separately.

Case a): $1 \leq \mu \leq \frac{1}{2\cos(\alpha-\beta)}$. Note that this inequality implies that the feasible set with respect to design variables α and β must satisfy $|\alpha-\beta| \geq \frac{\pi}{3}$, as shown in Figure 2. Our task in this case is to maximize d_2. Using (10), we have $\max d_2 = \max_{(\alpha,\beta)\in\Omega_1\cup\Omega_2}\max_{\mu}\frac{2P_{\mathrm{T}}}{F(\alpha,\beta,\mu)} = \frac{2P_{\mathrm{T}}}{\min_{(\alpha,\beta)\in\Omega_1\cup\Omega_2}\min_{\mu}F(\alpha,\beta,\mu)} = \frac{2P_{\mathrm{T}}}{\min_{(\alpha,\beta)\in\Omega_1\cup\Omega_2}F_1(\alpha,\beta)}$, where we have utilized the fact that $\min_{\mu}F(\alpha,\beta,\mu) = F(\alpha,\beta,1)$, since $1 \le \mu \le \frac{1}{2\cos(\alpha-\beta)}$ and $F(\alpha,\beta,\mu)$ is an increasing function of μ.

Case b): $\mu \ge \max\left\{\frac{1}{2\cos(\alpha-\beta)}, 2\cos(\alpha-\beta)\right\}$. This inequality infers that the feasible set of α and β must meet the inequality: $|\alpha-\beta| \ge \frac{\pi}{3}$ and $\theta_L \le \alpha, \beta \le \theta_U$ or $|\alpha-\beta| \le \frac{\pi}{3}$ and $\theta_L \le \alpha, \beta \le \theta_U$, as shown in Figure 2. Our goal in this case is to maximize d_2. With (10), we obtain

$$\begin{aligned}\max d_2 &= \max\Big\{\max_{(\alpha,\beta)\in\Omega_1\cup\Omega_2}\max_{\mu}\frac{2P_{\mathrm{T}}}{F(\alpha,\beta,\mu)},\\ &\qquad \max_{(\alpha,\beta)\in\Omega_3}\max_{\mu}\frac{2P_{\mathrm{T}}}{F(\alpha,\beta,\mu)}\Big\}\\ &= \max\Big\{\frac{2P_{\mathrm{T}}}{\min_{(\alpha,\beta)\in\Omega_1\cup\Omega_2}\min_{\mu}F(\alpha,\beta,\mu)},\\ &\qquad \frac{2P_{\mathrm{T}}}{\min_{(\alpha,\beta)\in\Omega_3}\min_{\mu}F(\alpha,\beta,\mu)}\Big\} \qquad (26)\end{aligned}$$

For any fixed α and β, since $F(\alpha,\beta,\mu)$ is an increasing function of μ, we have

$$\min_{(\alpha,\beta)\in\Omega_3}\min_{\mu}F(\alpha,\beta,\mu) = \min_{(\alpha,\beta)\in\Omega_3}F_2(\alpha,\beta) \qquad (27a)$$

and

$$\begin{aligned}&\min_{(\alpha,\beta)\in\Omega_1\cup\Omega_2}\min_{\mu}F(\alpha,\beta,\mu)\\ &= \min_{(\alpha,\beta)\in\Omega_1\cup\Omega_2}F\Big(\alpha,\beta,\frac{1}{2\cos(\alpha-\beta)}\Big)\\ &\ge \min_{(\alpha,\beta)\in\Omega_1\cup\Omega_2}F_1(\alpha,\beta) \qquad (27b)\end{aligned}$$

Combining (26) with (27) results in

$$\max d_2 \le \frac{2P_{\mathrm{T}}}{\min\Big\{\min_{(\alpha,\beta)\in\Omega_1\cup\Omega_2}F_1(\alpha,\beta), \min_{(\alpha,\beta)\in\Omega_3}F_2(\alpha,\beta)\Big\}}$$

Case c): $1 \le \mu \le 2\cos(\alpha-\beta)$. In this case, the feasible set in terms of α and β is $\Omega_3 = \{(\alpha,\beta) : |\alpha-\beta| \le \frac{\pi}{3}, \theta_L \le \alpha, \beta \le \theta_U\}$, as shown in Figure 2, and thus, we aim at maximizing d_3. Combining the definition of d_3 and (10), we can represent d_3^2 as $d_3^2 = d_2^2(\mu^2 - 2\mu\cos(\alpha-\beta)+1) = \frac{4P_{\mathrm{T}}^2\left(\mu^2-2\mu\cos(\alpha-\beta)+1\right)}{F^2(\alpha,\beta,\mu)} = \frac{4P_{\mathrm{T}}^2}{\Delta^2(\lambda_1,\lambda_2,\psi)}\frac{\mu^2-2\mu\cos(\alpha-\beta)+1}{f^2(\alpha,\beta,\mu)}$. Hence, we have $\max d_3^2 = \max_{\alpha,\beta}\max_{\mu} d_3^2$. Now, applying Lemma 5 into function $\frac{\mu^2-2\mu\cos(\alpha-\beta)+1}{f^2(\alpha,\beta,\mu)}$ yields $\max_{\mu} d_3^2 = \frac{4P_{\mathrm{T}}^2}{\Delta^2(\lambda_1,\lambda_2,\psi)}\frac{1}{f^2(\alpha,\beta,2\cos(\alpha,\beta))} = \frac{4P_{\mathrm{T}}^2}{F^2(\alpha,\beta,2\cos(\alpha,\beta))} = \frac{4P_{\mathrm{T}}^2}{F_2^2(\alpha,\beta)}$. Therefore, $\max d_3 = \frac{2P_{\mathrm{T}}}{\min_{(\alpha,\beta)\in\Omega_3}F_2(\alpha,\beta)}$.

Now, summarizing the above discussions on Cases a), b) and c), we conclude that the optimum is reached when $d_2 = d_3$ and is given by

$$\frac{2P_{\mathrm{T}}}{\min\Big\{\min_{(\alpha,\beta)\in\Omega_1\cup\Omega_2}F_1(\alpha,\beta), \min_{(\alpha,\beta)\in\Omega_3}F_2(\alpha,\beta)\Big\}}$$

On the other hand, by Lemmas 3 and 4, we know that $\min F_1(\alpha,\beta) = \{F_1(C), F_1(H), F_1(G)\}$, since $F_1(A) = F_1(C), F_1(H) = F_1(E)$ and $F_1(G) = F_1(F)$, and that $\min F_2(\alpha,\beta) = \{F_2(B), F_2(D), F_2(H), F_2(G), F_2(E), F_2(F)\}$. To further simplify this optimization problem, let us consider the case when $0 \le \psi \le \frac{\pi}{4}$. In this situation, we notice that $f_1(\alpha,\theta_U) = \cos(\alpha-\theta) + \cos(\theta_U-\theta)$, where $\theta_L \le \alpha \le \theta_U - \frac{\pi}{3}$. by the definition of θ in Lemma 2 and by Lemma 2, we know that $\theta_L \le \theta$ and $\theta_U - \frac{\pi}{3} < \theta$. As a result, we have $f_1(\theta_L,\theta_U) \le f_1(\theta_U-\frac{\pi}{3},\theta_U)$ and thus, $F_1(C) \le F_1(G)$. In addition, we also note that $f_1(\theta_L,\beta) = \cos(\theta_L-\theta) + \cos(\beta-\theta)$, where $\theta_L + \frac{\pi}{3} \le \beta \le \theta_U$. Hence, if $\theta_L + \frac{\pi}{3} \ge \theta$, then, $f_1(\theta_L,\theta_L+\frac{\pi}{3}) \ge f_1(\theta_L,\theta_U)$ and thus, we have $F_1(H) \ge F_1(C)$. If $\theta_L + \frac{\pi}{3} < \theta$, then, by Lemma 2, we know $\phi + \theta - \theta_L > \frac{\pi}{3}$. In addition, we also know that $f_2(\theta_L,\beta) = \sqrt{\left(2+\cos 2(\theta_L-\theta)\right)^2 + \sin^2 2(\theta_L-\theta)}\cos(\beta-\phi-\theta)$, where $\theta_L \le \beta \le \theta_L + \frac{\pi}{3}$. As a result, we obtain $f_2(\theta_L,\theta_L) < f_2(\theta_L,\theta+l+\frac{\pi}{3})$ and thus, $F_2(D) < F_2(H)$. Therefore, in any case, the point H is not any optimal point. Since $F_2(H) = F_1(H)$ and $F_2(F) = F_1(G)$, we have $\min\{\min F_1(\alpha,\beta), \min F_2(\alpha,\beta)\} = \{F_1(C), F_2(D)\}$. Now, by Lemma 6, we can prove the result on Theorem 1 regarding the considered case. Similarly, we can prove the case when $\frac{\pi}{4} < \psi \le \frac{\pi}{2}$. This completes the proof of Theorem 1. □

2) Proof of Theorem 2: Since $\mathbf{F} = \mathbf{V}\mathbf{G}$, the nonnegative constraint on $\mathbf{F}$ is transfered to $\mathbf{G}$ such that

$$\begin{aligned} g_{11}\cos\psi - g_{21}\sin\psi &\ge 0 \qquad (28a)\\ g_{11}\sin\psi + g_{21}\cos\psi &\ge 0 \qquad (28b)\\ g_{12}\cos\psi - g_{22}\sin\psi &\ge 0 \qquad (28c)\\ g_{12}\sin\psi + g_{22}\cos\psi &\ge 0 \qquad (28d)\end{aligned}$$

From (28) we can attain $g_{11} \ge 0$ and $g_{12} \ge 0$. In addition, when $0 \le \psi \le \frac{\pi}{4}$, from (28b) and (28d) we have

$$\begin{aligned} g_{21} &\ge -g_{11}\tan\psi \qquad (29a)\\ g_{22} &\ge -g_{12}\tan\psi \qquad (29b)\end{aligned}$$

and when $\frac{\pi}{4} < \psi \le \frac{\pi}{2}$, from (28a) and (28c) we obtain

$$\begin{aligned} g_{21} &\le g_{11}\cot\psi \qquad (30a)\\ g_{22} &\le g_{12}\cot\psi \qquad (30b)\end{aligned}$$

On the other hand, the nonnegative power constraint (4b) in terms of $\mathbf{F}$ is transfered into that in terms of $\mathbf{G}$ such that

$$\begin{aligned}&(\cos\psi + \sin\psi)(g_{11}+g_{12})\\ &+(\cos\psi - \sin\psi)(g_{21}+g_{22}) = 2P_{\mathrm{T}} \qquad (31)\end{aligned}$$

Now, combining (29) with (31) results in the largest feasible domain in terms of design variables g_{11} and g_{12}, i.e.,

$$g_{11} + g_{12} \le 2P_{\mathrm{T}}\cos\psi \quad \text{for } g_{11} \ge 0, g_{12} \ge 0 \qquad (32)$$

when $0 \le \psi \le \frac{\pi}{4}$, while combining (30) with (31) produces the largest feasible domain in terms of design variables g_{11} and g_{12}, i.e.,

$$g_{11} + g_{12} \le 2P_{\mathrm{T}}\sin\psi \quad \text{for } g_{11} \ge 0, g_{12} \ge 0 \qquad (33)$$

when $\frac{\pi}{4} < \psi \le \frac{\pi}{2}$. In addition, notice that in this case, from (5) we know that $g_{11} = d_1/\lambda_1, g_{12} = d_2/\lambda_1$ and $a_{12} = \lambda_1^2 g_{11}g_{12} = d_1 d_2$. Hence, by the definition of d_3, we obtain $d_3 = d_1 - d_2$ and thus, the objective function in Problem 2 is reduced to $\max\min\{d_2, d_1 - d_2\}$. This optimization problem can be split into two sub-problems: $\max d_2, d_1 \ge 2d_2$ and

$\max(d_1 - d_2), d_1 \leq 2d_2$ subject to either constraint (32) for $0 \leq \psi \leq \frac{\pi}{4}$, which is equivalent to $d_1 + d_2 \leq 2\lambda_1 P_{\mathrm{T}} \cos\psi$, or constraint (33) for $\frac{\pi}{4} < \psi \leq \frac{\pi}{2}$, which is equivalent to $d_1 + d_2 \leq 2\lambda_1 P_{\mathrm{T}} \sin\psi$. No mater which case occurs, the resulting optimization problem is linear and thus, the maximum is achieved when all the corresponding equalities hold, i.e., when $0 \leq \psi \leq \frac{\pi}{4}$, we have $d_1 = 2d_2, d_1 + d_2 = 2\lambda_1 P_{\mathrm{T}} \cos\psi$ and the equality in (29) holds, and when $\frac{\pi}{4} < \psi \leq \frac{\pi}{2}$, we have $d_1 = 2d_2, d_1 + d_2 = 2\lambda_1 P_{\mathrm{T}} \cos\psi$ and the equality in (30) holds. This completes the proof of Theorem 2. $\square$