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An Optimal QR Decomposition

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Abstract–The QR decomposition is a commonly used tool in various signal processing applications. The QR decomposition of a matrix H is a factorization $\mathrm{H} = \mathrm{QR}$, where Q is a unitary matrix and R is an upper triangular matrix. In this paper we propose an optimal QR decomposition, which we call as *QRS-decomposition*, $\mathrm{HS} = \mathrm{QR}$, where S is a unitary matrix. We shall show that the optimal matrix S is one that delivers an upper triangular matrix R whose diagonal entries are all *equal* to each other.

1. Introduction

The QR decomposition [1] is a commonly used tool in various signal processing applications. In multiple-input and multiple-output (MIMO) multiuser detection theory [2], the QR decomposition can be used to form the back-cancellation detector. The QR decomposition of a matrix $\mathbf{H}$ is a factorization $\mathbf{H} = \mathbf{QR}$, where $\mathbf{Q}$ is a unitary matrix and $\mathbf{R}$ is an upper triangular matrix. In this paper we propose an optimal QR decomposition, $\mathbf{HS} = \mathbf{QR}$, where $\mathbf{S}$ is a unitary matrix. We shall show that the optimal matrix $\mathbf{S}$ is one that delivers an upper triangular matrix $\mathbf{R}$ whose diagonal entries are all *equal* to each other. We call this special matrix $\mathbf{R}$ the *equal-diagonal R-factor*, and the resulting decomposition $\mathbf{HS} = \mathbf{QR}$, the *QRS-decomposition*.

There are two reasons to explain why we call such QRS decomposition is optimal? We will show that the decomposition has two important properties for communication signal processing:

1. The minimum Euclidean distance of the signal lattice before the superchannel $\mathbf{HS}$ is equals to the minimum Euclidean distance of the signal lattice after the superchannel up to the diagonal entry of $\mathbf{R}$ in the QRS decomposition of $\mathbf{H}$. We know that the free distance determines the detection performance of the maximum likelihood detector when the signal to noise ratio is high. Therefore, for the optimal precoder $\mathbf{S}$, the detection performance of the QR successive cancellation detector is asymptotically equivalent to that of the maximum likelihood detector as SNR $\to \infty$.

2. *The optimal order of colums in* $\mathbf{HS}$. Namely, among all column vectors of $\mathbf{HS}$, Arbitrary k-th column vector of $\mathbf{HS}$ has the minimal projection onto the space spanned by all the column vectors before the k-th column. Using this property we show that the optimal detection order in the VBLAST detector [3] on a channel that has the equal-diagonal is the natural order.

Notation: Notation $\mathbf{A}_k$ denotes a matrix consisting of the first k columns of matrix $\mathbf{A}$, i.e., $\mathbf{A}_k = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_k]$. By convention, $\mathbf{A}_0 = 1$. The remaining matrix after deleting columns $\mathbf{a}_{k_1}, \mathbf{a}_{k_2}, \cdots, \mathbf{a}_{k_i}$ is denoted by $\overline{\mathbf{A}}_{k_1,k_2\cdots,k_i}$. The j-th diagonal entry of a matrix $\mathbf{A}$ is denoted by $[\mathbf{A}]_j = A_{j,j}$. Notation $\mathbf{A}^\perp$ denotes the orthonormal complement of a matrix $\mathbf{A}$ in C^N. Notation $\mathbf{A}^+$ stands for the pseudo-inverse of $\mathbf{A}$. The transpose of $\mathbf{A}$ is denoted by $\mathbf{A}^T$. The Hermitian transpose of $\mathbf{A}$ (i.e., the conjugate and transpose of $\mathbf{A}$) is denoted by $\mathbf{A}^H$.

2. Review of the QR decomposition for successive cancellation detection

We first briefly review the successive cancellation detection algorithm that uses the QR decomposition, then we review the optimally ordered detector developed by Golden et al. [3], and finally we show how to equivalently represented this detector as a precoded QR-decomposition cancellation detector.

Let $\mathbf{x} = [x_1, \cdots, x_N]^T$ be an $N \times 1$ vector of symbols to be transmitted over a noisy channel. Each symbol x_i is chosen from a finite-size alphabet $\mathcal{X}$. Consider a general multiple-input and multiple-output (MIMO) channel model

$$\mathbf{r} = \mathbf{Hx} + \xi, \tag{1}$$

where $\mathbf{H}$ is an $M \times N$ full rank channel matrix (known to the receiver) with $M \geq N$, $\xi = [\xi_1, \cdots, \xi_M]^T$ is a white Gaussian noise vector where $E(\xi^H\xi) = \sigma^2\mathbf{I}$, and $\mathbf{r} = [r_1, \cdots, r_M]^T$ is the observed received vector. Our task is to detect (estimate) the vector $\mathbf{x} \in \mathcal{X}^N$ given the noisy observation $\mathbf{r}$. We denote the estimate of $\mathbf{x}$ by $\hat{\mathbf{x}} = [\hat{x}_1, \cdots, \hat{x}_N]^T$.

2.1. Successive cancellation detection using QR decomposition

The QR-decomposition-based successive cancellation detector is captured by the following three steps:

Algorithm 1 (QR-decomposition-based successive cancellation):

1. *QR-decomposition.* Perform the QR-decomposition, $\mathbf{H} = \mathbf{Q}\mathbf{R}$, where $\mathbf{Q}$ is a tall $M \times N$ column-wise orthonormal matrix and $\mathbf{R}$ is an upper triangular square matrix,

$$\mathbf{R} = \begin{pmatrix} R_{1,1} & R_{1,2} & \dots & R_{1,N} \\ 0 & R_{2,2} & \dots & R_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_{N,N} \end{pmatrix}.$$

Left-multiplying (1) by $\mathbf{Q}^H$, we get where $\tilde{\mathbf{r}} = [\tilde{r}_1, \cdots, \tilde{r}_N]^T = \mathbf{Q}^H \mathbf{r}$ and $\tilde{\xi} = [\tilde{\xi}_1, \cdots, \tilde{\xi}_N]^T = \mathbf{Q}^H \xi$. Equation (**??**) is equivalently written as

$$\tilde{r}_k = [\mathbf{R}]_k x_k + \sum_{m=k+1}^{N} R_{k,m} x_m + \tilde{\xi}_k,$$

where $[\mathbf{R}]_k$ denotes the k-th diagonal entry of $\mathbf{R}$.

2. *Hard decision.* We first estimate the symbol x_N by making the minimum-error-probability hard decision $\hat{x}_N = \mathrm{Quant}\left[\tilde{r}_N / [\mathbf{R}]_N\right]$. The function $q = \mathrm{Quant}(t)$ sets q to the element of $\mathcal{X}$ that is closest (in terms of Euclidean distance) to t.

3. *Cancellation.* Substitute the estimated symbol $\hat{x}_N$ back into the $(N-1)$-th row so as to remove the interference term in $\tilde{r}_{N-1}$ and then estimate x_{N-1}. Continue this procedure until we obtain the estimate of the first symbol x_1. The above procedure is described by the following recursive algorithm,

$$\begin{aligned} \hat{x}_N &= \mathrm{Quant}\left[\frac{\tilde{r}_N}{[\mathbf{R}_N]}\right] \\ \hat{x}_k &= \mathrm{Quant}\left[\frac{\tilde{r}_k - \sum_{m=k+1}^{N} R_{k,m}\hat{x}_m}{[\mathbf{R}]_k}\right] \end{aligned}$$

for $k = N-1, N-2, \cdots, 1$.

2.2. Optimally ordered detection

Golden et al. [3] proposed a vertical Bell Laboratories layered space-time (V-BLAST) system with an optimal ordered detection algorithm that maximizes the SNR.

Algorithm 2 (see Golden et al. [3]):

1. *Initial nulling.* Find an initial nulling vector with the smallest norm using zero-forcing. That is, find the index k, as the position of the smallest diagonal entry of $\left(\mathbf{H}^H\mathbf{H}\right)^{-1}$,

$$k_1 = \arg \min_{1 \le j \le N} \left[\left(\mathbf{H}^H\mathbf{H}\right)^{-1}\right]_j. \tag{2}$$

Then, project the received signal $\mathbf{r}$ onto the nulling direction and perform the hard decision to detect the symbol x_{k_1}. That is, set $\hat{x}_{k_1} = \mathrm{Quant}\left[\left(\mathbf{e}_{k_1}^{(0)}\right)^H \mathbf{r}\right]$, where $\mathbf{e}_{k_1}^{(0)}$ is the k_1-th column of $\mathbf{E}^{(0)} = \left(\mathbf{H}^+\right)^H$.

2. *Cancellation.* Subtract the detected signal from the received signal to get

$$\mathbf{r}^{(1)} = \mathbf{r} - \mathbf{h}_{k_1}\hat{x}_{k_1}, \tag{3}$$

where $\mathbf{h}_{k_1}$ denotes the k_1-th column of $\mathbf{H}$.

3. *Recursion.* Repeat the above two steps until all symbols are detected,

$$\begin{aligned} k_{i+1} &= \arg \min_{1 \le j \le N-i} \left[\left(\overline{\mathbf{H}}_{k_1,\cdots,k_i}^H \overline{\mathbf{H}}_{k_1,\cdots,k_i}\right)^{-1}\right]_j, \\ \hat{x}_{k_{i+1}} &= \mathrm{Quant}\left[\left(\mathbf{e}_{k_{i+1}}^{(i)}\right)^H \mathbf{r}^{(i)}\right], \\ \mathbf{r}^{(i+1)} &= \mathbf{r}^{(i)} - \mathbf{h}_{k_{i+1}}\hat{x}_{k_{i+1}} \end{aligned}$$

for $i = 1, 2, \cdots, N-1$, where $\mathbf{e}_{k_{i+1}}^{(i)}$ denotes the k_{i+1}-th column of $\mathbf{E}^{(i)} = \left(\mathbf{H}_{k_1,\cdots,k_i}^+\right)^H$ and $\mathbf{h}_{k_{i+1}}$ denotes the k_{i+1}-th column of $\mathbf{H}$.

2.3. QR interpretation of Algorithm 2

We use the QR decomposition to explain the algorithm of Golden et al. [3]. The first step (2) is equivalent to finding a subchannel whose SNR is the highest among all N possible subchannels. If we look at this problem from the viewpoint of the signal space that is spanned by the column vectors of $\mathbf{H}$, then the first step (2) is equivalent to finding a column vector of $\mathbf{H}$ whose projection onto the subspace spanned by all other column vectors is the smallest. Repeat the above procedure for the remaining columns. Finally, Algorithm 2 actually finds the optimal order. That is, it finds a permutation matrix $\mathbf{P} = [\mathbf{p}_{k_1}, \cdots, \mathbf{p}_{k_N}]$, where $\mathbf{p}_i$ denotes an $N \times 1$ vector whose i-th element is one, but others are zeros, such that the QR decomposition of $\mathbf{HP}$ gives rise to the optimally ordered successive cancellation detector.

Algorithm 3 (QR interpretation of Algorithm 2):

1. *Initialization.* Find the column vector of $\mathbf{H}$ whose projection onto the space spanned by all other column vectors is the smallest,

$$k_N = \arg \min_{1 \le k \le N} \left\| \left(\mathbf{I} - \overline{\mathbf{H}}_k \overline{\mathbf{H}}_k^+\right) \mathbf{h}_k \right\|^2.$$

Let $\alpha_N = \left(\mathbf{I} - \overline{\mathbf{H}}_{k_N}\overline{\mathbf{H}}_{k_N}^+\right)\mathbf{h}_{k_N}$ and $\mathbf{q}_N = \alpha_N / \parallel \alpha_N \parallel$.

2. *Recursion.* Repeat the first step by trimming the column vector one by one

$$\begin{aligned} k_i = \arg \min_{1 \le k \le N, k \ne k_{i+1},\cdots,k_N} &\left\| (\mathbf{I} - \overline{\mathbf{H}}_{k,k_{i+1},\cdots,k_N} \right. \\ &\left. \times \overline{\mathbf{H}}_{k,k_{i+1},\cdots,k_N}^+)\mathbf{h}_k \right\|^2 \end{aligned}$$

for $i = N-1, N-2, \cdots, 1$. Let

$$\alpha_i = \left(\mathbf{I} - \overline{\mathbf{H}}_{k_i,k_{i+1},\cdots,k_N}\overline{\mathbf{H}}^{+}_{k_i,k_{i+1},\cdots,k_N}\right)\mathbf{h}_{k_i}$$

and let $\mathbf{q}_i = \alpha_i / \parallel \alpha_i \parallel^2$.

3. *Permutation matrix formulation*. Finally, we obtain the optimal matrix $\mathbf{P} = [\mathbf{p}_{k_1}, \cdots, \mathbf{p}_{k_N}]$ and $\mathbf{Q} = [\mathbf{q}_1, \cdots, \mathbf{q}_N]$ such that $\mathbf{HP} = \mathbf{QR}$.

It is easily verified that Algorithm 2 is equivalent to applying Algorithm 1 to $\mathbf{r} = \mathbf{H}'\mathbf{x}' + \xi$, where $\mathbf{H}' = \mathbf{HP}$ and $\mathbf{x} = \mathbf{P}\mathbf{x}'$. Therefore, if we precode a vector $\mathbf{x}'$ with the permutation matrix $\mathbf{P}$, and apply Algorithm 1 to detect $\mathbf{x}'$, we get the optimally ordered successive-cancellation detector of Golden et al. [3]. In the remainder of the paper, we do *not* confine the precoder matrix to be a permutation matrix, and we derive the optimal QR decomposition.

3. QRS decomposition

In this section we develop the QRS decomposition and show its two important properties.

3.1. QRS decomposition

Theorem 1 *(QRS decomposition) For an arbitrary $M \times N$ matrix $\mathbf{H}$ with rank r, there exists a unitary matrix $\mathbf{S}$ such that $\mathbf{HS}$ has an equal-diagonal R-factor, i.e.*

$$\mathbf{HS} = \mathbf{QR}, \tag{4}$$

where $\mathbf{Q}$ is an $M \times r$ column-wise orthonormal matrix and $\mathbf{R} = [\ \mathbf{R}_{r\times r} \quad \mathbf{0}_{r\times(N-r)}\ ]$ with $\mathbf{R}_{r\times r}$ being the equal-diagonal R-factor.

3.2. Construction of the S-factor

The following recursive algorithm is to find the S-factor of the QRS decomposition $\mathbf{HS} = \mathbf{QR}$.

Algorithm 4 (Construction of the S-factor):

1. *SVD*. Perform the SVD of $\mathbf{H} = \mathbf{U\Lambda V}$ and form $\widetilde{\mathbf{H}} = \mathbf{U\Lambda}_r$.

Initialization. Determine the first row of $\widetilde{\mathbf{S}}$, i.e., $\widetilde{\mathbf{s}}_1 = (\widetilde{S}_{1,1}, \cdots, \widetilde{S}_{r,1})^T$, such that constraints

$$\begin{aligned} \widetilde{\mathbf{s}}_1^H \widetilde{\mathbf{H}}^H \widetilde{\mathbf{H}} \widetilde{\mathbf{s}}_1 &= det\left(\widetilde{\mathbf{H}}^H\widetilde{\mathbf{H}}\right)^{1/r} && (5)\\ \widetilde{\mathbf{s}}_1^H \widetilde{\mathbf{s}}_1 &= 1 && (6)\end{aligned}$$

are satisfied.

2. *Recursion* (reduce the dimension and decouple constraints). Set $\widetilde{\mathbf{s}}_{k+1} = \widetilde{\mathbf{S}}_k^{\perp}\mathbf{z}_{k+1}$, where $\mathbf{z}_{k+1}$ is any vector that satisfies

$$\begin{aligned} \mathbf{z}_{k+1}^H \mathbf{C}^{(k)} \mathbf{z}_{k+1} &= det\left(\widetilde{\mathbf{H}}^H\widetilde{\mathbf{H}}\right)^{1/r} && (7)\\ \mathbf{z}_{k+1}^H \mathbf{z}_{k+1} &= 1, && (8)\end{aligned}$$

with

$$\mathbf{C}^{(k)} = \left(\widetilde{\mathbf{S}}_k^{\perp}\right)^H \widetilde{\mathbf{H}}^H\widetilde{\mathbf{H}}\left(\mathbf{I} - \widetilde{\mathbf{S}}_k\left(\widetilde{\mathbf{S}}_k^H\widetilde{\mathbf{H}}^H\widetilde{\mathbf{H}}\widetilde{\mathbf{S}}_k\right)^{-1}\widetilde{\mathbf{S}}_k^H\widetilde{\mathbf{H}}^H\widetilde{\mathbf{H}}\right)\widetilde{\mathbf{S}}_k^{\perp}.$$

3. *Complete the S-factor*. $\mathbf{S} = [\mathbf{V}_r\widetilde{\mathbf{S}}, \overline{(\mathbf{V}^H)}_{1,\cdots,r}]$.

3.3. An explicit S-factor

In the above subsection we established that the solution for the S-factor is not unique. In the following we show how to find an explicit special solution of equations (5)-(8).

Algorithm 5 (Construction of an explicit S-factor):

1. *SVD*. Perform the SVD of $\mathbf{H} = \mathbf{U\Lambda V}$ and form $\widetilde{\mathbf{H}} = \mathbf{U\Lambda}_r$.

Initialization. An explicit solution for the first column of $\widetilde{\mathbf{S}}$, i.e., $\widetilde{\mathbf{s}}_1 = (\widetilde{S}_{1,1}, \cdots, \widetilde{S}_{r,1})^T$, is

$$\begin{aligned} \widetilde{S}_{1,1} &= \sqrt{\frac{det(\widetilde{\mathbf{H}}^H\widetilde{\mathbf{H}})^{1/r} - \lambda_r}{\lambda_1 - \lambda_r}}\\ \widetilde{S}_{k,1} &= 0 \qquad \text{for } k = 2, \cdots, r-1\\ \widetilde{S}_{r,1} &= \sqrt{\frac{\lambda_1 - det(\widetilde{\mathbf{H}}^H\widetilde{\mathbf{H}})^{1/r}}{\lambda_1 - \lambda_r}}.\end{aligned}$$

2. *Form the positive definite matrix for recursion*. Set (starting initially with $k = 1$)

$$\begin{aligned} \mathbf{C}^{(k)} &= \left(\widetilde{\mathbf{S}}_k^{\perp}\right)^H \widetilde{\mathbf{H}}^H\widetilde{\mathbf{H}}\\ &\times \left(\mathbf{I} - \widetilde{\mathbf{S}}_k\left(\widetilde{\mathbf{S}}_k^H\widetilde{\mathbf{H}}^H\widetilde{\mathbf{H}}\widetilde{\mathbf{S}}_k\right)^{-1}\widetilde{\mathbf{S}}_k^H\widetilde{\mathbf{H}}^H\widetilde{\mathbf{H}}\right)\widetilde{\mathbf{S}}_k^{\perp}.\end{aligned}$$

3. *Eigenvalue decomposition*. Perform the eigenvalue decomposition of $\mathbf{C}^{(k)}$,

$$\mathbf{C}^{(k)} = \mathbf{V}^{(k)}\Lambda^{(k)}\left(\mathbf{V}^{(k)}\right)^H,$$

where $\mathbf{V}^{(k)}$ is a $(r-k)\times(r-k)$ unitary matrix and $\Lambda^{(k)} = diag(\lambda_1^{(k)}, \cdots, \lambda_{r-k}^{(k)})$ with $\lambda_1^{(k)} \geq \lambda_2^{(k)} \geq \cdots \geq \lambda_{r-k}^{(k)} > 0$.

4. *Recursion*. Set $\widetilde{\mathbf{s}}_{k+1} = \widetilde{\mathbf{S}}_k^{\perp}\mathbf{V}^{(k)}\mathbf{y}^{(k)}$ for $k = 1, \cdots, r-2$, where $\mathbf{y}^{(k)} = \left[y_1^{(k)}, y_2^{(k)}, \cdots, y_{r-k}^{(k)}\right]^T$ is determined by

$$\begin{aligned} y_1^{(k)} &= \sqrt{\frac{det(\widetilde{\mathbf{H}}^H\widetilde{\mathbf{H}})^{1/r} - \lambda_{r-k}^{(k)}}{\lambda_1^{(k)} - \lambda_{r-k}^{(k)}}}\\ y_\ell^{(k)} &= 0 \qquad \text{for } \ell = 2, \cdots, r-k-1\\ y_{r-k}^{(k)} &= \sqrt{\frac{\lambda_1^{(k)} - det(\widetilde{\mathbf{H}}^H\widetilde{\mathbf{H}})^{1/r}}{\lambda_1^{(k)} - \lambda_{r-k}^{(k)}}}.\end{aligned}$$

When $k = r-1$, set $\widetilde{\mathbf{s}}_r = \widetilde{\mathbf{S}}_{r-2}^{\perp}\mathbf{V}^{(r-2)}\mathbf{y}^{(r-1)}$ where

$$\mathbf{y}^{(r-1)} = \begin{pmatrix} -\sqrt{\frac{det(\widetilde{\mathbf{H}}^H\widetilde{\mathbf{H}})^{1/r}-\lambda_2^{(r-2)}}{\lambda_1^{(r-2)}-\lambda_2^{(r-2)}}} \\ \sqrt{\frac{\lambda_1^{(r-2)}-det(\widetilde{\mathbf{H}}^H\widetilde{\mathbf{H}})^{1/r}}{\lambda_1^{(r-2)}-\lambda_2^{(r-2)}}} \end{pmatrix}.$$

5. *Complete the explicit S-factor*.

$$\mathbf{S} = [(\mathbf{V}^H)_r\widetilde{\mathbf{S}}, (\overline{\mathbf{V}^H})_{1,\cdots,r}]. \tag{9}$$

3.4. Properties of equal diagonal R-factors

Definition 1 *Define the minimum distance of the constellation $\mathcal{X}$ as*

$$d_{\min}(\mathcal{X}) = \min_{x\neq x', x,x'\in\mathcal{X}} |x-x'| = \sqrt{\min_{\mathbf{x},\mathbf{x}'\in\mathcal{X}^N, \mathbf{x}\neq\mathbf{x}'} ||\mathbf{x}-\mathbf{x}'||^2}$$

Definition 2 *Define the free distance of an $M\times N$ channel matrix $\mathbf{H}$ as*

$$d_{\text{free}}(\mathbf{H}) = \sqrt{\min_{\mathbf{x},\mathbf{x}'\in\mathcal{X}^N, \mathbf{x}\neq\mathbf{x}'} (\mathbf{x}-\mathbf{x}')^H\mathbf{H}^H\mathbf{H}(\mathbf{x}-\mathbf{x}')}$$

The following theorem shows that the free distance can be bounded in terms of the diagonal entries of the R-factor in the QR decomposition of a channel matrix.

Definition 3 *The column $\mathbf{h}_j$ of a channel $\mathbf{H}$ is called the* weakest link *if $\mathbf{h}_j$ is orthonormal to all previous columns, i.e., $\mathbf{h}_i^H\mathbf{h}_j = 0$ for $i = 1, \cdots, j-1$. The corresponding subscript j is called the* weakest link index. *The* weakest link index set *$\mathcal{J}$ consists of all weakest link indexes, i.e.,*

$$\mathcal{J} = \{j : \mathbf{h}_j \textit{ is the weakest link}\}.$$

Note that $\mathcal{J}$ is non-empty since $\mathbf{h}_1$ is always a weakest link. For every weakest link index j, we define the weakest link difference set

$$\mathcal{X}^{(j)} = \{\mathbf{x}-\mathbf{x}' : \mathbf{x}, \mathbf{x}' \in \mathcal{X}^N \textit{ and only the } j\textit{-th elements in } \mathbf{x} \textit{ and } \mathbf{x}' \textit{ are different}\}$$

Property 1 *Let $\mathbf{H}$ be an $M\times N$ full rank tall matrix that has an equal-diagonal R-factor. Then,*

$$d_{\text{free}}(\mathbf{H}) = \left(det\left(\mathbf{H}^H\mathbf{H}\right)\right)^{1/2N} d_{\min}(\mathcal{X}) = [\mathbf{R}]_k \cdot d_{\min}(\mathcal{X}). \tag{10}$$

Furthermore, $(\mathbf{x}-\mathbf{x}')^H\mathbf{H}^H\mathbf{H}(\mathbf{x}-\mathbf{x}') = d_{\text{free}}(\mathbf{H})$ if and only if the difference signal $\mathbf{x}-\mathbf{x}'$ of two transmitted vectors $\mathbf{x}$ and $\mathbf{x}'$ belongs to some weakest link difference set, i.e., $\mathbf{x}-\mathbf{x}' \in \mathcal{X}^{(j)}$ for a $j \in \mathcal{J}$.

Property 1 shows that for a channel matrix $\mathbf{H}$ with an equal-diagonal R-factor, the minimum Euclidean distance of the signal lattice before the channel ($d_{\min}(\mathcal{X})$) and the minimum Euclidean distance of the signal lattice after the channel ($d_{\text{free}}(\mathbf{H})$), satisfy relationship (10). We know that the free distance determines the detection performance of the maximum likelihood detector when the signal to noise ratio is high. From Property 1 we conclude that if the channel $\mathbf{H}$ has an equal-diagonal R-factor, then the free distance is computed by the QR decomposition. Therefore, this suggests that if the channel matrix has an equal-diagonal R-factor, the detection performance of the QR successive cancellation detector is asymptotically equivalent to that of the maximum likelihood detector as SNR $\to \infty$. In addition, Property 1 indicates in which positions of a block the free distance is reached, i.e., which positions are the weakest links. This may be exploited to design an encoder so as to increase the free distance of the encoded signal, which is well beyond the scope of this paper.

Suppose we wish to use the VBLAST detector [3] on a channel that has the equal-diagonal R-factor, a natural question is what is the optimal detection order? The following property gives the answer.

Property 2 *If a channel matrix has the equal-diagonal R-factor, the optimal detection order (that ensures that the high SNR components are detected first) is the natural order, i.e., $x_N \to x_{N-1} \to \cdots \to x_1$, in other words, the i-th symbol to be detected is the symbol x_{N+1-i}.*

Property 2 essentially characterizes a geometric property of a channel with the equal-diagonal R-factor. Namely, among all column vectors of $\mathbf{H}$, the last column vector $\mathbf{h}_N$ has the minimal projection onto the space spanned by all the other column vectors. Equivalently, $\mathbf{h}_N$ has the maximal distance from the space spanned by all the remaining column vectors. After we have eliminated $\mathbf{h}_N$, among all remaining column vectors of $\mathbf{H}$, the second to last column vector $\mathbf{h}_{N-1}$ has the maximum distance from the space spanned by all the remaining column vectors (except $\mathbf{h}_N$). We continue this procedure until we reach the first column vector $\mathbf{h}_1$.

References

[1] R. A. Horn and C. R. Johsson, *Matrix Analysis*. Cambridge University Press, 1993.

[2] S. Verdu, *Multiuser Detection*. Cambridge University Press, 1998.

[3] G. D. Golden, G. J. Foschini, R. A. Valenzuela and P.-W. Wolniansky, “Detection algorithm and initial laboratory results uisng V-BLAST space-time communication architecture,” *Electronics Letters*, vol. 35, no. 1, pp. 14-15, Jan. 7, 1999.