PROJECT: THE BLAHUT-ARIMOTO ALGORITHM

EE 4TM4: Digital Communications II

I. THE ALGORITHM

Let \( p(y|x) \) be a discrete memoryless channel with input alphabet \( \mathcal{X} \) and output alphabet \( \mathcal{Y} \). Recall that the capacity of \( p(y|x) \) is given by

\[
C = \max_{p(x)} I(X; Y).
\]

For any distribution \( p(x) \) on \( \mathcal{X} \) and any conditional distribution \( q(x|y) \) on \( \mathcal{X} \times \mathcal{Y} \), define

\[
f(p(x), q(x|y)) = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x) p(y|x) \log \frac{q(x|y)}{p(x)}.
\]

Prove the following facts:

1) The conditional distribution \( q^*(x|y) \) that maximizes \( f(p(x), q(x|y)) \) is given by

\[
q^*(x|y) = \frac{p(x)p(y|x)}{\sum_{x' \in \mathcal{X}} p(x')p(y|x')}.
\]

2) The distribution \( p^*(x) \) that maximizes \( f(p(x), q(x|y)) \) is given by

\[
p^*(x) = \frac{\prod_{y \in \mathcal{Y}} q(x|y)p(y|x)}{\sum_{x' \in \mathcal{X}} \prod_{y \in \mathcal{Y}} q(x'|y)p(y|x')}.
\]

3) The capacity of \( p(y|x) \) can be expressed as

\[
C = \max_{p(x)} \max_{q(y|x)} f(p(x), q(x|y)).
\]

The above facts naturally suggest the following iterative algorithm (known as the Blahut-Arimoto algorithm) for computing the channel capacity as well as the capacity-achieving input distribution. Let \( p^{(0)}(x) = \frac{1}{|\mathcal{X}|} \) for all \( x \in \mathcal{X} \). For \( k \geq 0 \), let

\[
q^{(k)}(x|y) = \frac{p^{(k)}(x)p(y|x)}{\sum_{x' \in \mathcal{X}} p^{(k)}(x')p(y|x')},
\]

\[
p^{(k+1)}(x) = \frac{\prod_{y \in \mathcal{Y}} q^{(k)}(x|y)p(y|x)}{\sum_{x' \in \mathcal{X}} \prod_{y \in \mathcal{Y}} q^{(k)}(x'|y)p(y|x')}.
\]

It can be shown that \( p^{(k)}(x) \) converges to a capacity-achieving distribution \( p^\dagger(x) \) (i.e., \( p^\dagger(x) \) is an optimal solution to the maximization problem in (1)) as \( k \to \infty \).

Remark: You are encouraged to consult the relevant literature on the Blahut-Arimoto algorithm.
II. ADDITIVE GAUSSIAN NOISE CHANNEL WITH PEAK POWER CONSTRAINT

Consider an additive Gaussian noise channel $Y = X + Z$, where $Z \sim \mathcal{N}(0, 1)$ is independent of $X$. Use the Blahut-Arimoto algorithm to compute the capacity-achieving distribution of this channel subject to the peak power constraint $|X| \leq A$ for $A = 0.1$, $A = 1$, and $A = 10$. Record and explain your findings.

Remark: You may need to discretize the input alphabet and the output alphabet in order to apply the Blahut-Arimoto algorithm.