WAVE PROPAGATION IN DIELECTRICS / CONDUCTORS : PHASOR FORM

SO FAR WE ASSUMED THAT THE EM ~~[illegible]~~ WAVE PROPAGATES IN FREE SPACE / PERFECT DIELECTRIC & THE WAVE IS PROPAGATING IN THE SOURCE-FREE ($J = 0$) REGION. LET US REMOVE THESE RESTRICTIONS & CONSIDER A GENERAL MEDIUM WHICH COULD BE A LOSSY DIELECTRIC OR A CONDUCTING MEDIUM.

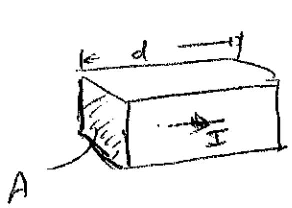
FROM AMPERE-MAXWELL'S LAW, WE HAVE

$$\nabla \times \vec{H} = \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t} \quad \rightarrow (1)$$

IN A CONDUCTOR, THERE ARE FREE ELECTRONS CONTRIBUTING TO J. FROM OHM'S LAW, WE HAVE

$$\vec{J} = \sigma \vec{E} \quad \rightarrow (2)$$

(QUICK PROOF :



$$J = \frac{I}{A} \; ; \quad E = \frac{V}{d}$$

$$I = \frac{V}{R}$$

$$JA = \frac{Ed}{R} \Rightarrow J = \underbrace{\left(\frac{d}{AR}\right)}_{\sigma} E$$

SINCE $R = \dfrac{d}{\sigma A}$)

USING (2) IN (1), WE FIND

$$\nabla \times \vec{H} = \sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t} \longrightarrow (3)$$

FARADAY'S LAW $\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \longrightarrow (4)$

LET $\vec{E} = E_x \vec{x}$ & $\vec{H} = H_y \vec{y}$

THE RHS OF (3) IS

$$\left(\sigma E_x + \epsilon \frac{\partial E_x}{\partial t} \right) \vec{x}.$$

SO, WE NEED TO CONSIDER ONLY THE X-COMPONENT OF $(\nabla \times H)$ IN (3)

$$(\nabla \times H)_x = -\frac{\partial H_y}{\partial z}$$

x y z x y

FROM (3), WE HAVE

$$\therefore -\frac{\partial H_y}{\partial z} = \sigma E_x + \epsilon \frac{\partial E_x}{\partial t} \longrightarrow (5)$$

THE RHS OF (4) IS

$$-\mu \frac{\partial H_y}{\partial t} \cdot \vec{y}$$

SO, WE NEED TO CONSIDER ~~only~~ ONLY THE y-COMPONENT OF $\nabla \times E$ IN 4.

$$(\nabla \times E)_y = +\frac{\partial E_x}{\partial z}$$

x y z x y

$$\therefore \frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} \longrightarrow (6)$$

WE ASSUME THAT THE FIELDS ARE HARMONICALLY VARYING WITH TIME ($e^{j\omega t}$) & EXPRESS THE FIELDS IN THE PHASOR FORM AS

$$E_x = Re\{E_{xs}\, e^{j\omega t}\}$$

$$H_y = Re\{H_{ys}\, e^{j\omega t}\}$$

HERE, E_{xs} & H_{ys} ARE THE PHASORS CORRESPONDING TO E_x & H_y RESPECTIVELY. NOTE THAT IF THE FIELDS ARE NOT CHANGING HARMONICALLY, THEY CAN BE EXPRESSED AS SUPERPOSITION ~~TO~~ OF $e^{j\omega_n t}$ (THANKS T. FOURIER). HOWEVER, THIS SUBJECT IS BEYOND THE SCOPE OF THIS COURSE.

NOTE: E_{xs} & H_{ys} ARE FUNCTIONS OF ~~SPACE~~ Z ONLY & THEY ARE INDEPENDENT OF TIME SINCE THE ~~TIME DEPEN~~ TIME-DEPENDENCE IS SEPARATED INTO THE FACTOR $e^{j\omega t}$.

WE HAVE SEEN BEFORE THAT, THE TERM $\frac{\partial E_x}{\partial t}$ TRANSLATES INTO $j\omega E_{xs}$ IN THE PHASOR WORLD. SO, EQS. (5) & (6) BECOME

$$-\frac{\partial H_{ys}}{\partial z} = \sigma E_{xs} + \epsilon j\omega E_{xs} \longrightarrow (7)$$

$$\frac{\partial E_{xs}}{\partial z} = -\mu j\omega H_{ys} \longrightarrow (8)$$

EQS. (7) & (8) ARE COUPLED. WE WISH TO FIND A SINGLE EQUATION FOR E_{xs} BY ELIMINATING H_{ys}. THIS CAN BE DONE BY DIFFERENTIATING EQ. (8) & MAKING USE OF EQ. (7):

$$\cancel{-\frac{\partial^2 H_y}{\partial z^2} = (\sigma + j\omega\epsilon)\frac{\partial E_{xs}}{\partial z} = (\sigma + j\omega\epsilon)(-\mu j\omega)H}$$

$$\frac{\partial^2 E_{xs}}{\partial z^2} = -\mu j\omega \frac{\partial H_{ys}}{\partial z}$$

$$= \mu j\omega(\sigma + j\epsilon\omega)\, E_{xs} \quad \text{(FROM (7))} \longrightarrow (9)$$

LET

$$k^2 = -j\omega\mu(\sigma + j\epsilon\omega) \longrightarrow (10)$$

k IS REFERRED AS THE COMPLEX PROPAGATION CONSTANT. NOW, EQ. (9) CAN BE WRITTEN AS

$$\frac{d^2 E_{xs}}{dz^2} = -k^2 E_{xs} \longrightarrow (11)$$

THIS IS THE SECOND ORDER ODE & ITS SOLUTION IS

$$E_{xs}(z) = A e^{jkz} + B e^{-jkz} \longrightarrow (12)$$

A & B ARE CONSTANTS TO BE DETERMINED FROM THE BOUNDARY CONDITIONS.

FREE SPACE / PERFECT DIELECTRIC:

TO MAKE CONNECTION WITH OUR PREVIOUS RESULTS ON FREE SPACE/PERFECT DIELECTRIC PROPAGATION, LET $\sigma = 0$ & ϵ BE REAL (NOTE: FOR A LOSSY DIELECTRIC, ϵ COULD BE COMPLEX). NOW, EQ. (10) BECOMES

$$k^2 = \omega^2 \mu \epsilon$$

SINCE, $v^2 = \dfrac{1}{\mu\epsilon}$ (v = SPEED OF EM WAVE)

$$k^2 = \frac{\omega^2}{v^2} = \beta^2 \text{ (SEE NOTES ON ~~FREE SPACE~~ PLANE WAVES)}$$

LET US FIRST CONSIDER THE FIRST TERM OF (12),

$$E_{xs} = A e^{jkz} = A e^{j\beta z} \; ; \quad \beta = \frac{\omega}{v}$$

SINCE

$$\begin{aligned} E_x &= Re\{E_{xs} e^{j\omega t}\} \\ &= Re\{A e^{j(\omega t + \beta z)}\} \\ &= A \cos(\omega t + \beta z) \end{aligned}$$

: BACKWARD ~~FORWARD~~ PROP. WAVE ~~WHEN~~ FOR $\beta > 0$.

NEXT CONSIDER THE SECOND TERM OF (12)

$$E_{xs} = B e^{-jkz}$$

$$\Rightarrow E_x = Re\{E_{xs} e^{j\omega t}\} = B \cos(\omega t - \beta z)$$

FORWARD PROP. WAVE.

SO, EQ. (12) REPRESENTS THE SUPERPOSITION OF PLANE WAVES PROPAGATING IN FORWARD & BACKWARD DIRECTIONS. TOTAL FIELD CAN BE WRITTEN AS

$$E_x = E^+_{x0} \cos(\omega t - \beta z) + E^-_{x0} \cos(\omega t + \beta z)$$

WHERE $E^+_{x0} = B =$ AMP. OF FORWARD PROP. WAVE

$E^-_{x0} = A =$ " " BACKWARD " "

~~LOSSY DIELECTRICS:~~