Chapter 1

Electromagnetics and Optics

1.1 Introduction

In this chapter, we will review the basics of electromagnetics and optics. We will briefly discuss various laws of electromagnetics leading to Maxwell's equations. The Maxwell's equations will be used to derive the wave equation which forms the basis for the study of optical fibers in Chapter 2. We will study elementary concepts in optics such as reflection, refraction and group velocity. The results derived in this chapter will be used throughout the book.

1.2 Coulomb's Law and Electric Field Intensity

In 1783, Coulomb showed experimentally that the force between two charges separated in free space or vacuum is directly proportional to the product of the charges and inversely proportional to the square of the distance between them. The force is repulsive if the charges are alike in sign, and attractive if they are of opposite sign, and it acts along the straight line connecting the charges. Suppose the charge q_1 is at the origin and q_2 is at a distance r as shown in Fig. 1.1. According to Coulomb's law, the force F_2 on the charge

q_2 is

$$\mathbf{F}_2 = \frac{q_1 q_2}{4\pi\epsilon r^2}\mathbf{r}, \tag{1.1}$$

where $\mathbf{r}$ is a unit vector in the direction of r and ϵ is called *permittivity* that depends on the medium in which the charges are placed. For free space, the permittivity is given by

$$\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{C^2/Nm^2} \tag{1.2}$$

For a dielectric medium, the permittivity, ϵ is larger than ϵ_0. The ratio of permittivity of a medium and permittivity of free space is called the relative permittivity, ϵ_r,

$$\frac{\epsilon}{\epsilon_0} = \epsilon_r \tag{1.3}$$

It would be convenient if we can find the force on a test charge located at any point in

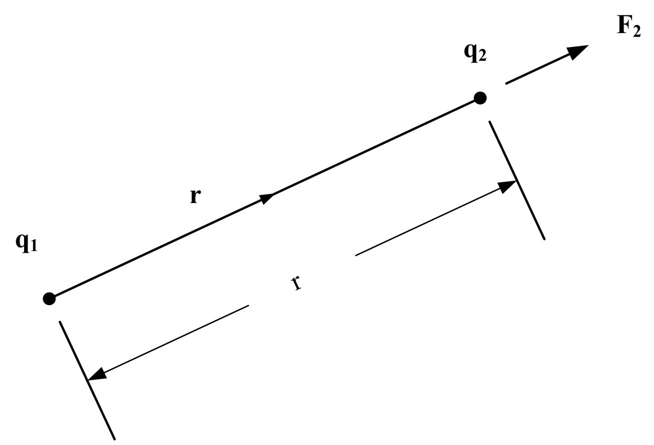

Figure 1.1. Force of attraction or repulsion between charges.

space due to a given charge q_1. This can be done by taking the test charge q_2 to be a unit positive charge. From Eq. (1.1), the force on the test charge is

$$\mathbf{E} = \mathbf{F}_2 = \frac{q_1}{4\pi\epsilon r^2}\mathbf{r} \tag{1.4}$$

The electric field intensity is defined as the force on a positive unit charge and is given by Eq. (1.4). The electric field intensity is a function only of the charge q_1 and the distance between the test charge and q_1.

For historical reasons, the product of electric field intensity and permittivity is defined as the electric flux density $\mathbf{D}$.

$$\mathbf{D} = \epsilon \mathbf{E} = \frac{q_1}{4\pi r^2}\mathbf{r} \tag{1.5}$$

Electric flux density is a vector with its direction same as the electric field intensity. Imagine a sphere S of radius r around the charge q_1 as shown in Fig. 1.2. Consider an incremental area ΔS on the sphere. The electric flux crossing this surface is defined as the product of the normal component of $\mathbf{D}$ and the area ΔS.

$$\text{Flux crossing } \Delta S = \Delta\psi = D_n \Delta S, \tag{1.6}$$

where D_n is the normal component of $\mathbf{D}$.The direction of the electric flux density is normal

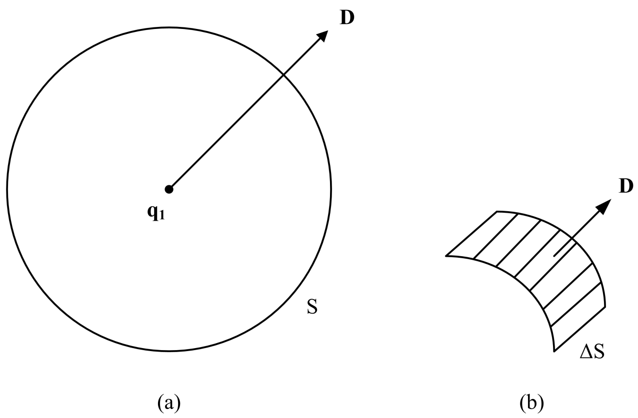

Figure 1.2. (a) Electric flux density on the surface of the sphere. (b) The incremental surface ΔS on the sphere.

to the surface of the sphere and therefore, from Eq. (1.5) we obtain $D_n = q_1/4\pi r^2$. If we add the differential contributions to flux from all the incremental surfaces of the sphere, we obtain the total electric flux passing through the sphere,

$$\psi = \int d\psi = \oint_S D_n dS \tag{1.7}$$

Since the electric flux density D_n given by Eq. (1.5) is the same at all the points on the surface of the sphere, the total electric flux is simply the product of D_n and surface area of the sphere $4\pi r^2$,

$$\psi = \oint_S D_n dS = \frac{q_1}{4\pi r^2} \times \text{surface area} = q_1 \tag{1.8}$$

Thus, the electric flux passing through a sphere is equal to the charge enclosed by the sphere. This is known as *Gauss's law.* Although we considered the flux crossing a sphere, Eq. (1.8) holds true for any arbitrary closed surface. This is because the surface element ΔS of an arbitrary surface may not be perpendicular to the direction of $\mathbf{D}$ given by Eq. (1.5) and the projection of the surface element of an arbitrary closed surface in a direction normal to D is the same as the surface element of a sphere. From Eq. (1.8), we see that the total flux crossing the sphere is independent of the radius. This is because the electric flux density is inversely proportional to the square of the radius while the surface area of the sphere is directly proportional to the square of the radius and therefore, total flux crossing a sphere is the same no matter what its radius is.

So far we have assumed that the charge is located at a point. Next, let us consider the case when the charge is distributed in a region. Volume charge density is defined as the ratio of the charge q and the volume element ΔV occupied by the charge as it shrinks to zero,

$$\rho = \lim_{\Delta V \to 0} \frac{q}{\Delta V} \tag{1.9}$$

Dividing Eq. (1.8) by ΔV where ΔV is the volume of the surface S and letting this volume to shrink to zero, we obtain

$$\lim_{\Delta V \to 0} \frac{\oint_S D_n dS}{\Delta V} = \rho \tag{1.10}$$

The left hand side of Eq. (1.10) is called *divergence* of $\mathbf{D}$ and is written as

$$\text{div } \mathbf{D} = \nabla \cdot \mathbf{D} = \lim_{\Delta V \to 0} \frac{\oint_S D_n dS}{\Delta V} \tag{1.11}$$

and Eq. (1.11) can be written as

$$\text{div } \mathbf{D} = \rho \tag{1.12}$$

The above equation is called the *differential form of Gauss's law* and it is the first of Maxwell's four equations. The physical interpretation of Eq. (1.12) is as follows. Suppose

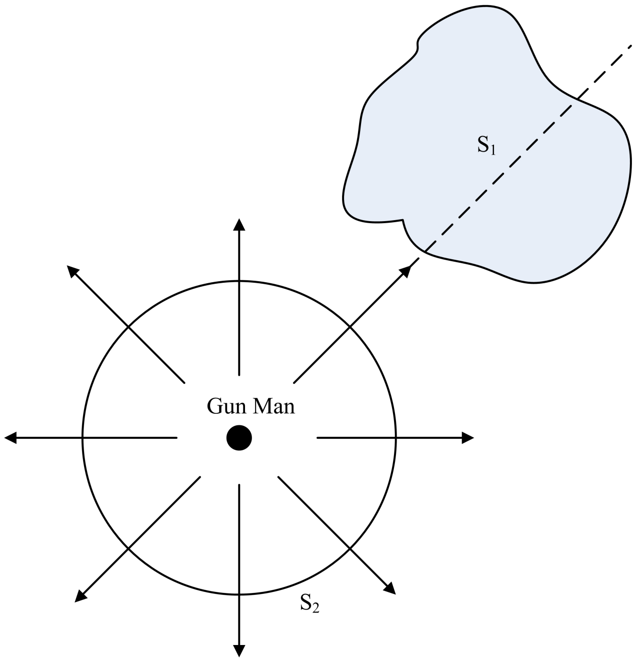

Figure 1.3. Divergence of bullet flow.

a gun man is firing bullets in all directions as shown in Fig. 1.3 [2]. Imagine a surface S_1 that does not enclose the gun man. The net outflow of the bullets through the surface S_1 is zero since the number of bullets entering this surface is the same as the number of bullets leaving the surface. In other words, there is no *source* or *sink* of bullets in the region S_1. In this case, we say that the divergence is zero. Imagine a surface S_2 that encloses the gun man. There is a net outflow of bullets since the gun man is the *source* of bullets who lies within the surface S_2 and divergence is not zero. Similarly, if we imagine a closed surface in a region that encloses charges with charge density ρ, the divergence is not zero and is given by Eq. (1.12). In a closed surface that does not enclose charges, the divergence is zero.

1.3 Ampere's Law and Magnetic Field Intensity

Consider a conductor carrying a direct current I. If we bring a magnetic compass near the conductor, it will orient in a direction shown in Fig. 1.4(a). This indicates that the magnetic needle experiences the magnetic field produced by the current. Magnetic field intensity **H** is defined as the force experienced by an isolated unit positive magnetic charge (Note that an isolated magnetic charge q_m does not exist without an associated $-q_m$) just like the electric field intensity, **E** is defined as the force experienced by a unit positive electric charge.

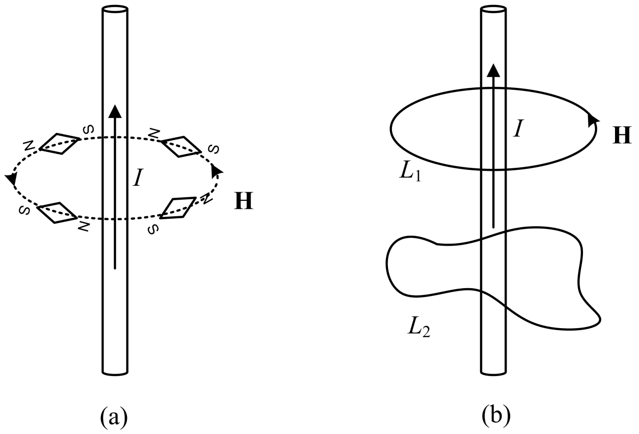

Figure 1.4. (a) Direct current-induced constant magnetic field. (b) Ampere's circuital law.

Consider a closed path L_1 or L_2 around the current-carrying conductor as shown in Fig. 1.4(b). Ampere's circuital law states that the line integral of **H** about any closed path is equal to the direct current enclosed by that path.

$$\oint_{L_1} \mathbf{H} \cdot d\mathbf{L} = \oint_{L_2} \mathbf{H} \cdot d\mathbf{L} = I \tag{1.13}$$

The above equation indicates that the sum of the components of **H** that are parallel to the tangent of a closed curve *times* the differential path length is equal to the current enclosed by this curve. If the closed path is a circle (L_1) of radius r, due to circular symmetry, magnitude of **H** is constant at any point on L_1 and its direction is shown in Fig. 1.4(b). From Eq. (1.13), we obtain

$$\oint_{L_1} \mathbf{H} \cdot d\mathbf{L} = H \times \text{circumference} = I \tag{1.14}$$

or

$$H = \frac{I}{2\pi r} \tag{1.15}$$

Thus, the magnitude of magnetic field intensity at a point is inversely proportional to its distance from the conductor. Suppose the current is flowing in z direction. The z-component of current density J_z may be defined as the ratio of the incremental current ΔI passing through an elemental surface area $\Delta S = \Delta X \Delta Y$ perpendicular to the direction of the current flow as the surface ΔS shrinks to zero,

$$J_z = \lim_{\Delta S \to 0} \frac{\Delta I}{\Delta S}. \tag{1.16}$$

Current density $\mathbf{J}$ is a vector with its direction given by the direction of current. If $\mathbf{J}$ is not perpendicular to the surface ΔS, we need to find the component J_n that is perpendicular to the surface by taking the dot product

$$J_n = \mathbf{J} \cdot \mathbf{n}, \tag{1.17}$$

where $\mathbf{n}$ is a unit vector normal to the surface ΔS. By defining a vector $\Delta \mathbf{S} = \Delta S \mathbf{n}$, we have

$$J_n \Delta S = \mathbf{J} \cdot \Delta \mathbf{S} \tag{1.18}$$

and incremental current ΔI is given by

$$\Delta I = \mathbf{J} \cdot \Delta \mathbf{S} \tag{1.19}$$

Total current flowing through a surface S is obtained by integrating,

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} \tag{1.20}$$

Using Eq. (1.20) in Eq. (1.13), we obtain

$$\oint_{L1} \mathbf{H} \cdot d\mathbf{L} = \int_S \mathbf{J} \cdot d\mathbf{S}, \tag{1.21}$$

where S is the surface whose perimeter is the closed path L_1.

In analogy with the definition of electric flux density, magnetic flux density is defined as

$$\mathbf{B} = \mu \mathbf{H} \tag{1.22}$$

where μ is called the *permeability.* In free space, the permeability has a value

$$\mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2 \tag{1.23}$$

In general, permeability of a medium μ is written as a product of the permeability of free space μ_0 and a constant that depends on the medium. This constant is called relative permeability μ_r.

$$\mu = \mu_0 \mu_r \tag{1.24}$$

The magnetic flux crossing a surface S can be obtained by integrating the normal component of magnetic flux density,

$$\psi_m = \int_S B_n dS \tag{1.25}$$

If we use the Gauss's law for the magnetic field, the normal component of the magnetic flux density integrated over a closed surface should be equal to the magnetic charge enclosed. However, no isolated magnetic charge has ever been discovered. In the case of electric field, the flux lines start from or terminate on electric charges. In contrast, magnetic flux lines are closed and do not emerge from or terminate on magnetic charges. Therefore,

$$\psi_m = \int_S B_n dS = 0 \tag{1.26}$$

and in analogy with the differential form of Gauss's law for electric field, we have

$$\text{div } \mathbf{B} = 0 \tag{1.27}$$

The above equation is one of Maxwell's four equations.

1.4 Faraday's Law

Consider an iron core with copper windings connected to a voltmeter as shown in Fig. 1.5. If you bring a bar magnet close to the core, you will see a deflection in the voltmeter. If you stop moving the magnet, there will be no current through the voltmeter. If you

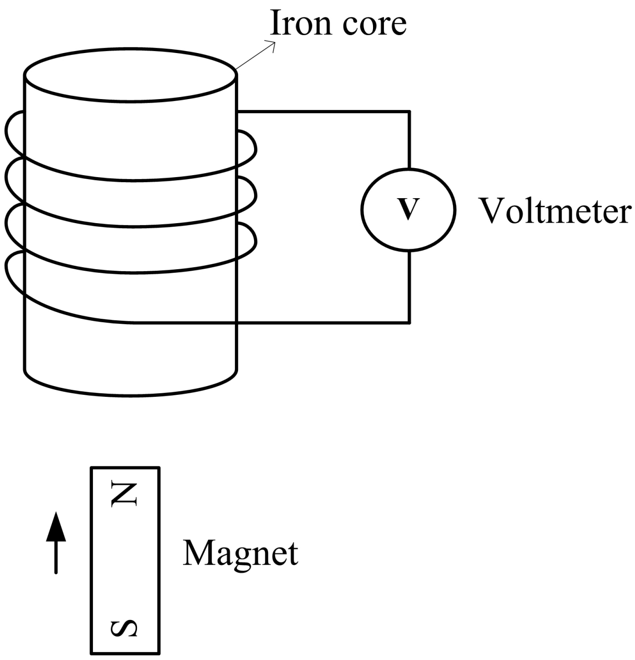

Figure 1.5. Generation of emf by moving a magnet.

move the magnet away from the conductor, the deflection of the voltmeter will be in the opposite direction. Same results can be obtained if the core is moving and the magnet is stationary. Faraday carried out an experiment similar to the one shown in Fig. 1.5 and from his experiments, he concluded that the time varying magnetic field produces an electromotive force which is responsible for a current in a closed circuit. An electromotive force is simply the electric field intensity integrated over the length of the conductor or in other words, it is the voltage developed. In the absence of electric field intensity, electrons move randomly in all directions with a zero net current in any direction. Because of the electric field intensity (which is the force experienced by a unit electric charge) due to time varying magnetic field, electrons are forced to move in a particular direction leading to current. Faraday's law is stated as

$$emf = -\frac{d\psi_m}{dt}, \tag{1.28}$$

where *emf* is the electromotive force about a closed path L (that includes conductor and connections to voltmeter), ψ_m is the magnetic flux crossing the surface S whose perimeter is the closed path L and $d\psi_m/dt$ is the time rate of change of this flux. Since emf is an integrated electric field intensity, it can be expressed as

$$emf = \oint_L \mathbf{E} \cdot d\mathbf{l} \tag{1.29}$$

Magnetic flux crossing the surface S is equal to the sum of the normal component of the magnetic flux density at the surface *times* the elemental surface area dS,

$$\psi_m = \int_S B_n dS = \int_S \mathbf{B} \cdot d\mathbf{S}, \tag{1.30}$$

where $d\mathbf{S}$ is a vector with its magnitude dS and its direction normal to the surface. Using Eqs. (1.29) and (1.30) in Eq. (1.28), we obtain

$$\begin{aligned} \oint_L \mathbf{E} \cdot d\mathbf{l} &= -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} \\ &= -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \end{aligned} \tag{1.31}$$

In Eq. (1.31), we have assumed that the path is stationary and the magnetic flux density is changing with time and therefore, the elemental surface area is not time dependent allowing us to take the partial derivative under the integral sign. In Eq. (1.31), we have a line integral on the left hand side and a surface integral on the right hand side. In vector calculus, a line integral could be replaced by a surface integral using Stokes' theorem,

$$\oint_L \mathbf{E}.d\mathbf{l} = \int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} \tag{1.32}$$

to obtain

$$\int_S \left[\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right] \cdot d\mathbf{S} = 0 \tag{1.33}$$

Eq. (1.33) is valid for any surface whose perimeter is a closed path. It holds true for any arbitrary surface only if the integrand vanishes, i.e.,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.34}$$

The above equation is Faraday's law in the differential form and is one of Maxwell's four equations.

1.4.1 Meaning of Curl

The curl of a vector $\mathbf{A}$ is defined as

$$curl\ \mathbf{A} = \nabla \times \mathbf{A} = F_x\mathbf{x} + F_y\mathbf{y} + F_z\mathbf{z} \tag{1.35}$$

$$F_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \tag{1.36}$$

$$F_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \tag{1.37}$$

$$F_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \tag{1.38}$$

Consider a vector $\mathbf{A}$ with only x-component. The z-component of the curl of $\mathbf{A}$ is

$$F_z = -\frac{\partial A_x}{\partial y} \tag{1.39}$$

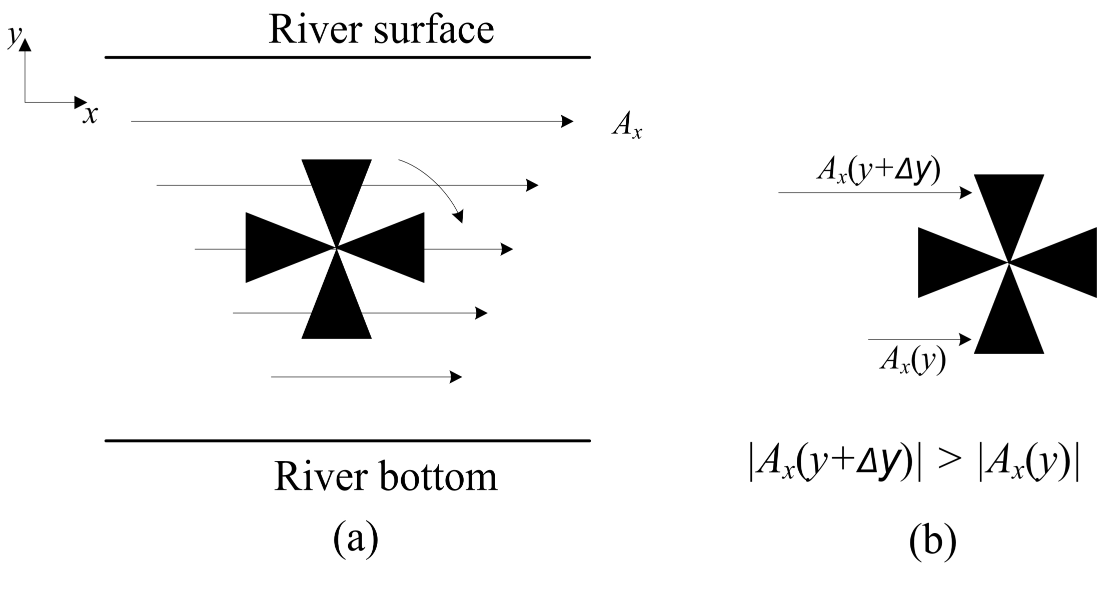

Figure 1.6. Clockwise movement of the paddle when the velocity of water increases from bottom to the surface of river.

Skilling [1] suggests the use of a paddle wheel to measure the curl of a vector. As an example, consider the water flow in a river as shown in Fig. 1.6(a). Suppose the velocity of water (A_x) increases as we go from the bottom of the river to the surface. The length of arrow in Fig. 1.6(a) represents the magnitude of the water velocity. If we place a paddle wheel with its axis perpendicular to the paper, it will turn clockwise since the upper paddle experiences more force than the lower paddle (Fig. 1.6(b)). In this case, we say that curl exists along the axis of the paddle wheel in a direction of an inward normal to the surface of the page (z direction). Larger speed of the paddle means larger value of the curl.

Suppose the velocity of water is the same at all depths as shown in Fig. 1.7. In this

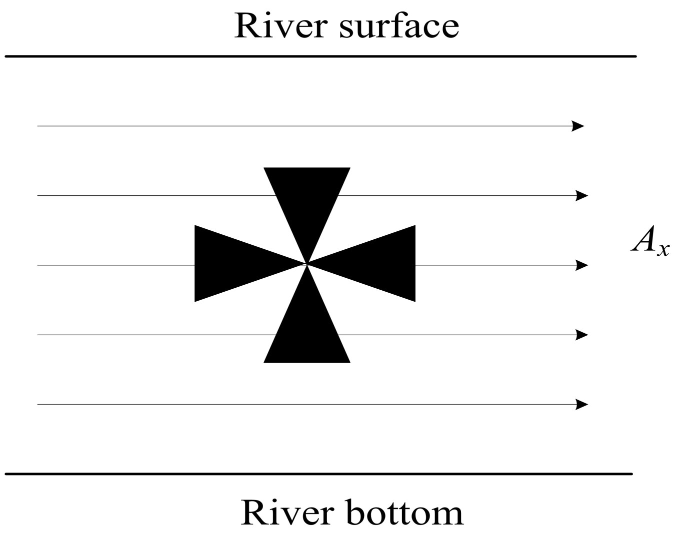

Figure 1.7. Velocity of water is constant at all depths. The paddle wheel does not rotate in this case.

case, the paddle wheel will not turn which means there is no curl in a direction of the axis of the paddle wheel. From Eq. (1.39), we find that the z-component of the curl is zero if the water velocity A_x does not change as a function of depth y.

Eq. (1.34) can be understood as follows. Suppose the x-component of the electric field intensity E_x is changing as a function of y in a conductor, as shown in Fig. 1.8. This implies that there is a curl perpendicular to the page. From Eq. (1.34), we see that this should be equal to the time derivative of the magnetic field intensity in z-direction. In other words, the time-varying magnetic field in the z-direction induces electric field intensity as shown in Fig. 1.8. The electrons in the conductor move in a direction opposite to E_x (Coulomb's law) leading to the current in the conductor if the circuit is closed.

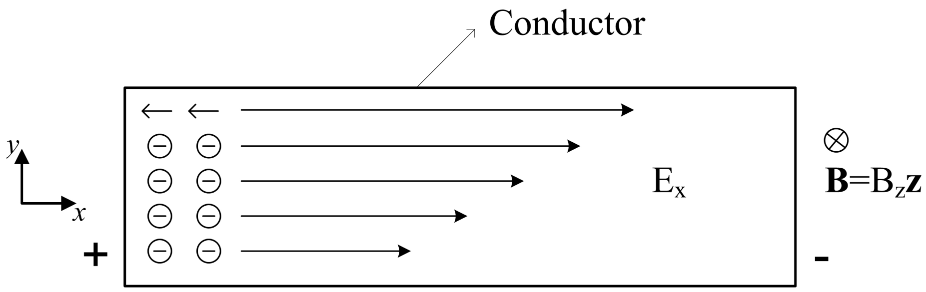

Figure 1.8. Induced electric field due to the time-varying magnetic field perpendicular to the page.

1.4.2 Ampere's Law in Differential Form

From Eq. (1.21), we have

$$\oint_{L_1} \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} \tag{1.40}$$

Using Stokes' theorem (Eq. (1.32)), Eq. (1.40) may be rewritten as

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S} \tag{1.41}$$

or

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{1.42}$$

The above equation is the differential form of Ampere's circuital law and it is one of Maxwell's four equations for the case of current and electric field intensity not changing with time. Eq. (1.40) holds true only under the non-time varying conditions. From Faraday's law (Eq. (1.34)), we see that if the magnetic field changes with time, it produces an electric field. Due to symmetry, one might expect that the time-changing electric field produces magnetic field. However, comparing Eqs. (1.34) and (1.42), we find that the term corresponding to time varying electric field is missing in Eq. (1.42). Maxwell proposed to add a term to the right hand side of Eq. (1.42) so that time-changing electric field produces magnetic field. With this modification, Ampere's circuital law becomes

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{1.43}$$

In the absence of the second term on the right hand side of Eq. (1.43), it can be shown that the law of conservation of charges is violated (See Problem 1.4). The second term is known as displacement current density.

1.5 Maxwell's Equations

Combining Eqs. (1.12),(1.27),(1.34) and (1.43), we obtain

$$\begin{aligned} \text{div } \mathbf{D} &= \rho, && (1.44) \\ \text{div } \mathbf{B} &= 0, && (1.45) \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, && (1.46) \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, && (1.47) \end{aligned}$$

From Eqs. (1.46) and (1.47), we see that time changing magnetic field produces electric field and time changing electric field or current density produces magnetic field. The charge distribution ρ and current density $\mathbf{J}$ are the *sources* for generation of electric and magnetic fields. For the given charge and current distribution, Eqs. (1.44)-(1.47) may be solved to obtain the electric and magnetic field distributions. The terms on the right hand sides of Eqs. (1.46) and (1.47) may be viewed as the sources for generation of field intensities appearing on the left hand sides of Eqs.(1.46) and (1.47). As an example, consider the alternating current $I_0 \sin(2\pi f t)$ flowing in the transmitter antenna. From Ampere's law, we find that the current leads to magnetic field intensity around the antenna (first term of Eq. (1.47)). From Faraday's law, it follows that the time-varying magnetic field induces electric field intensity (Eq. (1.46)) in the vicinity of the antenna. Consider a point in the neighborhood of antenna (but not on the antenna). At this point

J=0, but the time-varying electric field intensity or displacement current density (second term on the right hand side of Eq.(1.47)) leads to magnetic field intensity, which in turn leads to electric field intensity (Eq.(1.46)). This process continues and the generated electromagnetic wave propagates outward just like the water wave generated by throwing a stone into a lake. If the displacement current density were to be absent, there would be no continuous coupling between electric and magnetic fields and we would not have had electromagnetic waves.

1.5.1 Maxwell's Equation in Source-Free Region

In free space or dielectric, if there is no charge or current in the neighborhood, we can set $\rho = 0$ and $J = 0$ in Eq. (1.44). Note that the above equations describe the relations between electric field, magnetic field and the sources at a space-time point and therefore, in a region sufficiently far away from the sources, we can set $\rho = 0$ and $J = 0$ in that region. However, on the antenna, we can not ignore the source terms ρ or J in Eqs. (1.44)-(1.47). Setting $\rho = 0$ and $J = 0$ in the source-free region, Maxwell's equations take the form

$$\begin{aligned} \text{div } \mathbf{D} &= 0, & (1.48)\\ \text{div } \mathbf{B} &= 0, & (1.49)\\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, & (1.50)\\ \nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t}, & (1.51) \end{aligned}$$

In the source-free region, time changing electric/magnetic field (which was generated from a distant source ρ or $\mathbf{J}$) acts as a *source* for magnetic/electric field.

1.5.2 Electromagnetic Wave

Suppose the electric field is only along x-direction,

$$\mathbf{E} = E_x \mathbf{x}, \qquad (1.52)$$

and magnetic field is only along y-direction,

$$\mathbf{H} = H_y \mathbf{y}. \tag{1.53}$$

Substituting Eqs. (1.52) and (1.53) into Eq. (1.50), we obtain

$$\nabla \times \mathbf{E} = \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{bmatrix} = \frac{\partial E_x}{\partial z}\mathbf{y} - \frac{\partial E_x}{\partial y}\mathbf{z} = -\mu \frac{\partial H_y}{\partial t}\mathbf{y}. \tag{1.54}$$

Equating y- and z-components separately, we find

$$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t}, \tag{1.55}$$

$$\frac{\partial E_x}{\partial y} = 0. \tag{1.56}$$

Substituting Eqs. (1.52) and (1.53) into Eq. (1.51), we obtain

$$\nabla \times \mathbf{H} = \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & H_y & 0 \end{bmatrix} = -\frac{\partial H_y}{\partial z}\mathbf{x} + \frac{\partial H_y}{\partial x}\mathbf{z} = \epsilon \frac{\partial E_x}{\partial t}\mathbf{x}. \tag{1.57}$$

Therefore,

$$\frac{\partial H_y}{\partial z} = -\epsilon \frac{\partial E_x}{\partial t}, \tag{1.58}$$

$$\frac{\partial H_y}{\partial x} = 0. \tag{1.59}$$

Eqs. (1.55) and (1.58) are coupled. To obtain an equation that does not contain H_y, we differentiate Eq. (1.55) with respect to z and differentiate Eq. (1.58) with respect to t,

$$\frac{\partial^2 E_x}{\partial z^2} = -\mu \frac{\partial H_y}{\partial t \partial z}, \tag{1.60}$$

$$\mu \frac{\partial^2 H_y}{\partial z \partial t} = -\mu\epsilon \frac{\partial^2 E_x}{\partial t^2}. \tag{1.61}$$

Adding Eqs. (1.60) and (1.61), we obtain

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\epsilon \frac{\partial E_x}{\partial t^2}. \tag{1.62}$$

The above equation is called the *wave equation* and it forms the basis for the study of electromagnetic wave propagation.

1.5.3 Free Space Propagation

For free space, $\epsilon = \epsilon_0 = 8.854 \times 10^{-12} C^2/Nm^2$, $\mu = \mu_0 = 4\pi \times 10^{-7} N/A^2$, and

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \simeq 3 \times 10^8 m/s, \tag{1.63}$$

where c is the velocity of light in free space. Before Maxwell's time, electrostatics, magnetostatics and optics were unrelated. Maxwell unified these three fields and showed that the light wave is actually an electromagnetic wave with its velocity given by Eq. (1.63).

1.5.4 Propagation in a Dielectric Medium

Similar to Eq. (1.63), velocity of light in a medium can be written as

$$v = \frac{1}{\sqrt{\mu \epsilon}}, \tag{1.64}$$

where $\mu = \mu_0 \mu_r$ and $\epsilon = \epsilon_0 \epsilon_r$. Therefore,

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0 \mu_r \epsilon_r}}. \tag{1.65}$$

Using Eq. (1.64) in Eq. (1.65), we have

$$v = \frac{c}{\sqrt{\mu_r \epsilon_r}}. \tag{1.66}$$

For dielectrics, $\mu_r = 1$ and velocity of light in a dielectric medium can be written as

$$v = \frac{c}{\sqrt{\epsilon_r}} = \frac{c}{n}, \tag{1.67}$$

where $n = \sqrt{\epsilon_r}$ is called the refractive index of the medium. The refractive index of a medium is greater than 1 and velocity of light in a medium is less than that in free space.

1.6 1-Dimensional Wave Equation

Using Eq. (1.64) in Eq. (1.62), we obtain

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 E_x}{\partial t^2}. \tag{1.68}$$

Elimination of E_x from Eqs. (1.55) and (1.58) leads to the same equation for H_y,

$$\frac{\partial^2 H_y}{\partial z^2} = \frac{1}{v^2}\frac{\partial H_y}{\partial t^2} \tag{1.69}$$

To solve Eq. (1.68), let us try a trial solution of the form

$$E_x(t, z) = f(t + \alpha z), \tag{1.70}$$

where f is an arbitrary function of $t + \alpha z$. Let

$$u = t + \alpha z \tag{1.71}$$

$$\frac{\partial u}{\partial z} = \alpha, \quad \frac{\partial u}{\partial t} = 1, \tag{1.72}$$

$$\frac{\partial E_x}{\partial z} = \frac{\partial E_x}{\partial u}\frac{\partial u}{\partial z} = \frac{\partial E_x}{\partial u}\alpha, \tag{1.73}$$

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{\partial^2 E_x}{\partial u^2}\alpha^2, \tag{1.74}$$

$$\frac{\partial^2 E_x}{\partial t^2} = \frac{\partial^2 E_x}{\partial u^2}. \tag{1.75}$$

Using Eqs. (1.74) and (1.75) in Eq. (1.68), we obtain

$$\alpha^2 \frac{\partial^2 E_x}{\partial u^2} = \frac{1}{v^2}\frac{\partial^2 E_x}{\partial u^2}. \tag{1.76}$$

Therefore,

$$\alpha = \pm\frac{1}{v}, \tag{1.77}$$

$$E_x = f\left(t + \frac{z}{v}\right) \text{ or } E_x = f\left(t - \frac{z}{v}\right) \tag{1.78}$$

The negative sign implies a forward propagating wave and the positive sign indicates a backward propagating wave. Note that f is an arbitrary function and it is determined by the initial conditions as illustrated by the following examples.

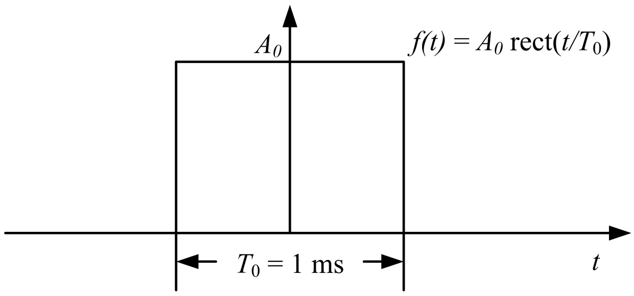

Figure 1.9. Electrical field $E_x(t, 0)$ at the flash light

Example 1.1

Turn on the flash light for 1 ms and turn it off. You will generate a pulse shown in Fig. 1.9 at the flash light ($z = 0$). The electric field intensity oscillates at light frequencies and the rectangular shape shown in Fig. 1.9 is actually the absolute field envelope. Let us ignore the fast oscillations in this example and write the field (which is actually the field envelope[1]) at $z = 0$ as

$$E_x(t, 0) = f(t) = A_0 \text{ rect}\left(\frac{t}{T_0}\right), \tag{1.79}$$

where

$$\text{rect}(x) = \begin{cases} 1, & \text{if} \quad |x| < 1/2 \\ 0, & \text{otherwise} \end{cases} \tag{1.80}$$

and $T_0 = 1$ ms. The speed of light in free space, $v = c \simeq 3 \times 10^8$m/s. Therefore, it takes

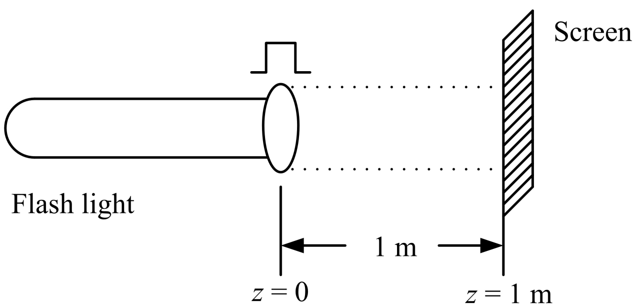

Figure 1.10. The propagation of the light pulse generated at the flash light.

[1]In can be shown that the field envelope also satisfies wave equation.

$0.33 \times 10^{-8} s$ to get the light pulse on the screen. At $z = 1$ m,

$$E_x(t, z) = f\left(t - \frac{z}{v}\right) = A_0 \text{rect}\left(\frac{t - 0.33 \times 10^{-8}}{T_0}\right). \tag{1.81}$$

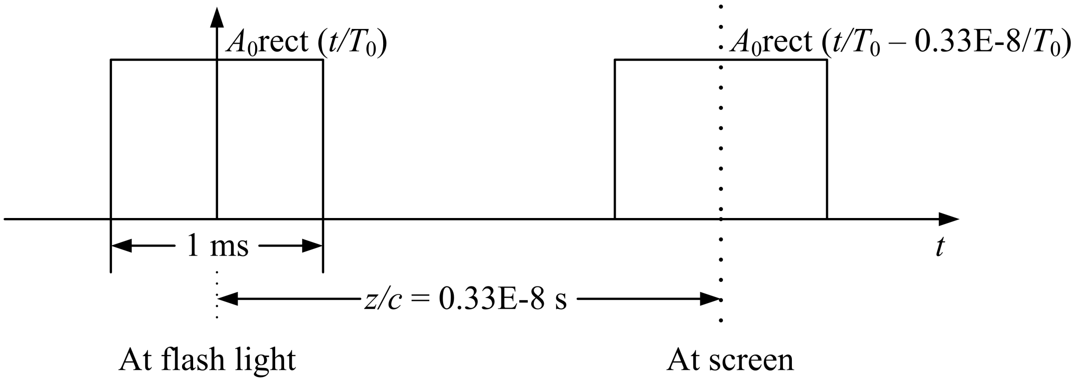

Figure 1.11. The electric field envelopes at the flash light and at the screen.

Example 1.2

A laser operates at 191 THz. Under ideal conditions and ignoring transverse distributions, the laser output may be written as

$$E_x(t, 0) = f(t) = A_0 \cos(2\pi f_0 t), \tag{1.82}$$

where $f_0 = 191$ THz. The laser output arrives at the screen after $0.33 \times 10^{-8} s$. The

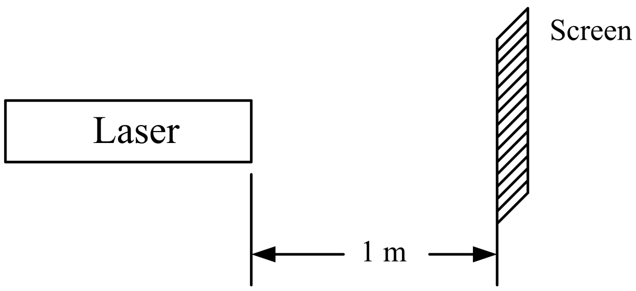

Figure 1.12. The propagation of laser output in free space.

electric field intensity at the screen may be written as

$$\begin{aligned} E_x(t,z) &= f\left(t-\frac{z}{v}\right) \\ &= A\cos\left[2\pi f_0\left(t-\frac{z}{v}\right)\right] \\ &= A\cos\left[2\pi f_0(t-0.33\times 10^{-8})\right]. \end{aligned} \tag{1.83}$$

Example 1.3

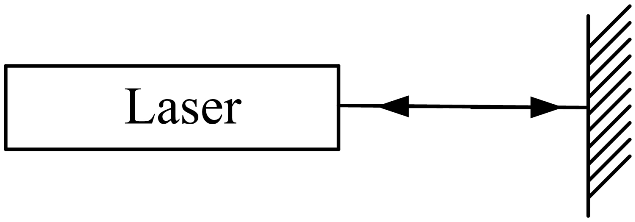

Figure 1.13. Reflection of the laser output by a mirror.

The laser output is reflected by a mirror and it propagates in backward direction as shown in Fig. 1.13. In Eq. (1.78), the positive sign corresponds to backward propagating wave. Suppose that at the mirror electromagnetic wave undergoes a phase shift of ϕ [2] . The backward propagating wave can be described by (see Eq. (1.78)

$$E_{x-} = A\cos[2\pi f_0(t+z/v)+\phi] \tag{1.84}$$

The forward propagating wave is described by Eq. (1.83) ,

$$E_{x+} = A\cos[2\pi f_0(t-z/v)] \tag{1.85}$$

Total field is given by

$$E_x = E_{x+} + E_{x-} \tag{1.86}$$

[2]If the mirror is a perfect conductor, $\phi = \pi$.

1.6.1 1-Dimensional Plane Wave

The output of the laser in Example 1.2 propagates as a *plane wave* as given by Eq. (1.83). A plane wave can be written in any of the following forms:

$$\begin{aligned} E_x(t,z) &= E_{x0}\cos\left[2\pi f\left(t-\frac{z}{v}\right)\right], \\ &= E_{x0}\cos\left[2\pi f t-\frac{2\pi}{\lambda}z\right], \\ &= E_{x0}\cos\left(\omega t-kz\right), \end{aligned} \tag{1.87}$$

where v is the velocity of light in the medium, f is the frequency, $\lambda = v/f$ is the wavelength, $\omega = 2\pi f$ is the angular frequency, $k = 2\pi/\lambda$ is the wave number, and k is also called the propagation constant. Frequency and wavelength are related by

$$v = f\lambda, \tag{1.88}$$

or equivalently

$$v = \frac{\omega}{k}. \tag{1.89}$$

Since H_y also satisfies the wave equation (Eq. (1.69)), it can be written as

$$H_y = H_{y0}\cos\left(\omega t-kz\right) \tag{1.90}$$

From Eq. (1.58), we have

$$\frac{\partial H_y}{\partial z} = -\epsilon\frac{\partial E_x}{\partial t}, \tag{1.91}$$

Using Eq. (1.87) in Eq. (1.91), we obtain

$$\frac{\partial H_y}{\partial z} = \epsilon\omega E_{x0}\sin\left(\omega t-kz\right) \tag{1.92}$$

Integrating Eq. (1.92) with respect to z,

$$H_y = \frac{\epsilon E_{x0}\omega}{k}\cos\left(\omega t-kz\right)+D \tag{1.93}$$

where D is a constant of integration and it could depend on t. Comparing Eqs. (1.90) and (1.93), we see that D is zero and using Eq. (1.89), we find

$$\frac{E_{x0}}{H_{y0}} = \frac{1}{\epsilon v} = \eta, \tag{1.94}$$

where η is the intrinsic impedance of the dielectric medium. For freespace $\eta = 376.47$ Ohms. Note that E_x and H_y are independent of x and y. In other words, at time t, the phase $\omega t - kz$ is constant in a transverse plane described by $z =$ constant and therefore, they are called plane waves.

1.6.2 Complex Notation

It is often convenient to use the complex notation for electric and magnetic fields in the following forms:

$$\widetilde{E}_x = E_{x0} e^{i(\omega t - kz)} \;\; \texttt{or} \;\; \widetilde{E}_x = E_{x0} e^{-i(\omega t - kz)} \tag{1.95}$$

and

$$\widetilde{H}_y = H_{y0} e^{i(\omega t - kz)} \;\; \texttt{or} \;\; \widetilde{H}_y = H_{y0} e^{-i(\omega t - kz)} \tag{1.96}$$

This is known as analytic representation. The actual electric and magnetic fields can be obtained by

$$E_x = \mathrm{Re}\left[\widetilde{E}_x\right] = E_{x0} \cos(\omega t - kz) \tag{1.97}$$

and

$$H_y = \mathrm{Re}\left[\widetilde{H}_y\right] = H_{y0} \cos(\omega t - kz) \tag{1.98}$$

In reality, the electric and magnetic fields are not complex, but we represent them in the complex forms of Eqs. (1.95) and (1.96) with the understanding that the real parts of the complex fields corresponds to the actual electric and magnetic fields. This representation leads to mathematical simplifications. For example, differentiation of a complex exponential function is the complex exponential function multiplied by some constant. In the analytic representation, superposition of two eletromagnetic fields corresponds to addition of two complex fields. However, care should be exercised when we take the product of two electromagnetic fields as encountered in nonlinear optics. For example, consider the product of two electrical fields given by

$$E_{xn} = A_n \cos(\omega_n t - k_n z), \quad n = 1, 2 \tag{1.99}$$

$$\begin{aligned} E_{x1}E_{x2} &= \frac{A_1 A_2}{2} \cos[(\omega_1 + \omega_2)t - (k_1 + k_2)z] + \\ &\quad \cos[(\omega_1 - \omega_2)t - (k_1 - k_2)z] \end{aligned} \tag{1.100}$$

The product of the electromagnetic fields in the complex forms is

$$\widetilde{E}_{x1}\widetilde{E}_{x2} = A_1 A_2 \exp[i(\omega_1 + \omega_2)t - i(k_1 + k_2)z] \tag{1.101}$$

If we take the real part of Eq. (1.101), we find

$$\begin{aligned} \mathrm{Re}\left[\widetilde{E}_{x1}\widetilde{E}_{x1}\right] &= A_1 A_2 \cos[(\omega_1 + \omega_2)t - (k_1 + k_2)z] \\ &\neq E_{x1}E_{x2} \end{aligned} \tag{1.102}$$

In this case, we should use the real form of electromagnetic fields. In the rest of the book, we sometimes omit ˜ and use $E_x(H_y)$ to represent complex electric (magnetic) field with the understanding that the real part is the actual field.

1.7 Power Flow and Poynting Vector

Consider an electromagnetic wave propagating in a region V with the cross-sectional area A as shown in Fig. 1.14. The propagation of a plane electromagnetic wave in the source free region are governed by Eqs. (1.58) and (1.60),

$$\epsilon \frac{\partial E_x}{\partial t} = -\frac{\partial H_y}{\partial z} \tag{1.103}$$

$$\mu \frac{\partial H_y}{\partial t} = -\frac{\partial E_x}{\partial z} \tag{1.104}$$

Multiplying Eq. (1.103) by E_x and noting that

$$\frac{\partial E_x^2}{\partial t} = 2E_x \frac{\partial E_x}{\partial t}, \tag{1.105}$$

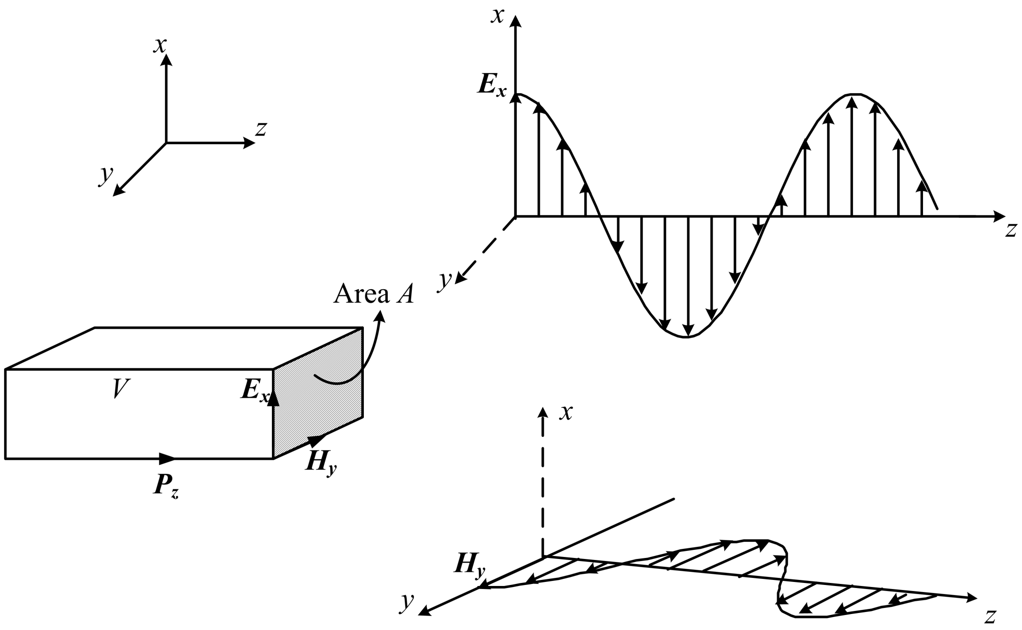

Figure 1.14. Electromagnetic wave propagation in a volume V with cross-sectional area A.

we obtain

$$\frac{\epsilon}{2}\frac{\partial E_x^2}{\partial t} = -E_x \frac{\partial H_y}{\partial z} \tag{1.106}$$

Similarly, multiplying Eq. (1.104) by H_y, we have

$$\frac{\mu}{2}\frac{\partial H_y^2}{\partial t} = -H_y \frac{\partial E_x}{\partial z} \tag{1.107}$$

Adding Eqs. (1.107) and (1.106) and integrating over the volume V, we obtain

$$\frac{\partial}{\partial t}\int_V \left[\frac{\epsilon E_x^2}{2} + \frac{\mu H_y^2}{2}\right] dV = -A\int_0^L \left[E_x \frac{\partial H_y}{\partial z} + H_y \frac{\partial E_x}{\partial z}\right] dz \tag{1.108}$$

On the right hand side of Eq. (1.108), integration over the transverse plane yields the area A since E_x and H_y are functions of z only. Eq. (1.108) can be rewritten as

$$\frac{\partial}{\partial t}\int_V \left[\frac{\epsilon E_x^2}{2} + \frac{\mu H_y^2}{2}\right] dV = -A\int_0^L \frac{\partial}{\partial z}\left[E_x H_y\right] dz = -AE_x H_y\Big|_0^L \tag{1.109}$$

The terms $\epsilon E_x^2/2$ and $\mu H_y^2/2$ represent *energy densities* of electric field and magnetic field, respectively. The left hand side of Eq. (1.109) can be interpreted as the power crossing the area A and therefore, $E_x H_y$ is the power per unit area or *power density* measured in watts per square meter (W/m^2). We define a Poynting vector $\mathcal{P}$ as

$$\mathcal{P} = \mathbf{E} \times \mathbf{H} \tag{1.110}$$

The z-component of the Poynting vector is

$$\mathcal{P}_z = E_x H_y \tag{1.111}$$

The direction of the Poynting vector is normal to both **E** and **H** vectors and in fact, it is the direction of power flow.

In Eq. (1.109), integrating the energy density over volume leads to energy, $\mathcal{E}$ and therefore, it can be rewritten as

$$\frac{1}{A}\frac{d\mathcal{E}}{dt} = \mathcal{P}_z(0) - \mathcal{P}_z(L) \tag{1.112}$$

The left hand side of (1.112) represents the rate of change of energy per unit area and therefore, $\mathcal{P}_z$ has the dimension of power per unit area or power density. For lightwaves, the power density is also known as *optical intensity*. Eq. (1.112) states that the difference in the power entering the corss-section A and power leaving the cross-section A is equal to the rate of change of energy in the volume V. The plane wave solutions for E_x and H_y are given by Eqs. (1.87) and (1.90),

$$\begin{aligned} E_x &= E_{x0} \cos(\omega t - kz) && (1.113) \\ H_y &= H_{y0} \cos(\omega t - kz) && (1.114) \\ \mathcal{P}_z &= \frac{E_{x0}^2}{\eta} \cos^2(\omega t - kz) && (1.115) \end{aligned}$$

The average power density may be found by integrating it over one cycle and divide by the period $T = 1/f$.

$$\begin{aligned} \mathcal{P}_z^{av} &= \frac{1}{T}\frac{E_{x0}^2}{\eta}\int_0^T \cos^2(\omega t - kz)\, dt, && (1.116) \\ &= \frac{1}{T}\frac{E_{x0}^2}{\eta}\int_0^T \frac{1 + \cos[2(\omega t - kz)]}{2} dt && (1.117) \\ &= \frac{E_{x0}^2}{2\eta}. && (1.118) \end{aligned}$$

The integral of the cosine function over one period is zero and therefore, the second term of Eq. (1.118) does not contribute after the integration. The average power density $\mathcal{P}_z^{av}$ is proportional to square of the electric field amplitude. Using complex notation, Eq. (1.111) can be written as

$$\begin{aligned}\mathcal{P}_z &= \mathrm{Re}\left[\widetilde{E}_x\right]\mathrm{Re}\left[\widetilde{H}_y\right] &(1.119)\\ &= \frac{1}{\eta}\mathrm{Re}\left[\widetilde{E}_x\right]\mathrm{Re}\left[\widetilde{E}_x\right] = \frac{1}{\eta}\left[\frac{\widetilde{E}_x+\widetilde{E}_x^*}{2}\right]\left[\frac{\widetilde{E}_x+\widetilde{E}_x^*}{2}\right] &(1.120)\end{aligned}$$

The right hand side of Eq. (1.120) contains the product terms such as $\widetilde{E}_x^2$ and $\widetilde{E}_x^{*^2}$. The average of E_x^2 and $E_x^{*^2}$ over the period T is zeros since they are sinusoids with no dc component. Therefore, the average power density is given by

$$\mathcal{P}_z^{av} = \frac{1}{2\eta T}\int_0^T \left|\widetilde{E}_x\right|^2 dt = \frac{\left|\widetilde{E}_x\right|^2}{2\eta}, \tag{1.121}$$

since $\left|\widetilde{E}_x\right|^2$ is a constant for the plane wave. Thus, we see that, in complex notation, the average power density is proportional to the absolute square of the amplitude.

Problem 1.1

Two monochromatic waves are superposed to obtain

$$\widetilde{E}_x = A_1\exp[i(\omega_1 t - k_1 z)] + A_2\exp[i(\omega_2 t - k_2 z)] \tag{1.122}$$

Find the average power density of the combined wave.

Solution From Eq. (1.121), we have

$$\begin{aligned}\mathcal{P}_z^{av} &= \frac{1}{2\eta T}\int_0^T \left|\widetilde{E}_x\right|^2 dt\\ &= \frac{1}{2\eta T}\Big\{T|A_1|^2 + T|A_2|^2 + A_1A_2^\star\int_0^T \exp[i(\omega_1-\omega_2)t - i(k_1-k_2)z]dt\\ &\quad +A_2A_1^\star\int_0^T \exp[-i(\omega_1-\omega_2)+i(k_1-k_2)z]]\Big\}dt &(1.123)\end{aligned}$$

Since integrals of sinusoids over the period T is zero, the last two terms in Eq. (1.123) do not contribute, which leads to

$$\mathcal{P}_z^{av} = \frac{|A_1|^2 + |A_2|^2}{2\eta} \tag{1.124}$$

Thus, the average power density is the sum of absolute squares of the amplitudes of monochromatic waves.

1.8 3-Dimensional Wave Equation

From Maxwell's equations, the following wave equation could be derived (See Problem 1.6)

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{\upsilon^2}\frac{\partial^2\psi}{\partial t^2} = 0 \tag{1.125}$$

where ψ is any one of components E_x, E_y, E_z, H_x, H_y, H_z. As before, let us try a trial solution of the form

$$\psi = f\left(t - \alpha_x x - \alpha_y y - \alpha_z z\right) \tag{1.126}$$

Proceeding as in section (1.6), we find that

$$\alpha_x^2 + \alpha_y^2 + \alpha_z^2 = \frac{1}{\upsilon^2} \tag{1.127}$$

If we choose the function to be a cosine function, we obtain a 3-dimensional plane wave described by

$$\begin{aligned} \psi &= \psi_0 \cos\left[\omega\left(t - \alpha_x x - \alpha_y y - \alpha_z z\right)\right] && (1.128) \\ &= \psi_0 \cos\left(\omega t - k_x x - k_y y - k_z z\right) && (1.129) \end{aligned}$$

where $k_r = \omega\alpha_r$, $r = x, y, z$. Define a vector $\mathbf{k} = k_x\mathbf{x} + k_y\mathbf{y} + k_z\mathbf{z}$. $\mathbf{k}$ is known as *wave vector*. Eq. (1.127) becomes

$$\frac{\omega^2}{k^2} = \upsilon^2 \text{ or } \frac{\omega}{k} = \pm\upsilon \tag{1.130}$$

where k is the magnitude of the vector $\mathbf{k}$,

$$k = \sqrt{k_x^2 + k_y^2 + k_z^2} \tag{1.131}$$

k is also known as *wave number*. The angular frequency ω is determined by the light source such as a laser or LED. In a linear medium, the frequency of the launched electromagnetic wave can not be changed. The frequency of the plane wave propagating in a medium of refractive index n is same as that of the source although the wavelength in the medium decreases by a factor n. For the given angular frequency ω, the wave number in a medium of refractive index n can be determined by

$$k = \frac{\omega}{v} = \frac{\omega n}{c} = \frac{2\pi n}{\lambda_0} \tag{1.132}$$

where $\lambda_0 = c/f$ is the free space wavelength. For free space, $n = 1$ and the wave number is

$$k_0 = \frac{2\pi}{\lambda_0} \tag{1.133}$$

The wavelength λ_m is a medium of refractive index n can be defined by

$$k = \frac{2\pi}{\lambda_m} \tag{1.134}$$

Comparing (1.132) and (1.134), it follows that

$$\lambda_m = \frac{\lambda_0}{n} \tag{1.135}$$

Example 1.4

Consider a plane wave propagating in x-z plane making an angle of 30^0 with z-axis. This plane wave may be described by

$$\psi = \psi_0 \cos\left(\omega t - k_x x - k_z z\right) \tag{1.136}$$

The wave vector $\mathbf{k} = k_x\mathbf{x} + k_z\mathbf{z}$. From the Fig. 1.15, $k_x = k\cos 60^0 = k/2$ and $k_z = k\cos 30^0 = k\sqrt{3}/2$. Eq. (1.136) may be written as

$$\psi = \psi_0 \cos\left[\omega t - k\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}z\right)\right] \tag{1.137}$$

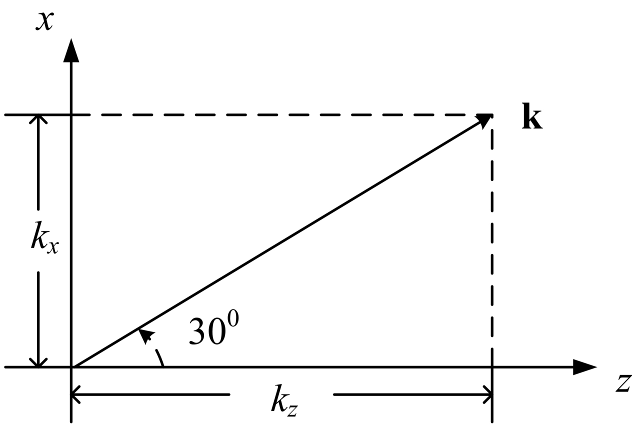

Figure 1.15. A plane wave propagates at angle 30° with z-axis.

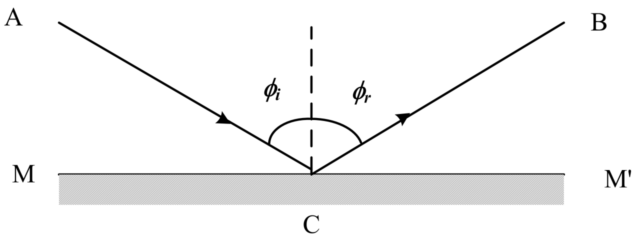

Figure 1.16. Reflection of a light wave incident on a mirror.

1.9 Reflection and Refraction

Reflection and refraction occur when light enters into a new medium with a different refractive index. Consider a ray incident on the mirror MM'. According to the law of reflection, the angle of reflection ϕ_r is equal to the angle of incidence ϕ_i

$$\phi_i = \phi_r$$

The above result can be proved from Maxwell's equations with appropriate boundary

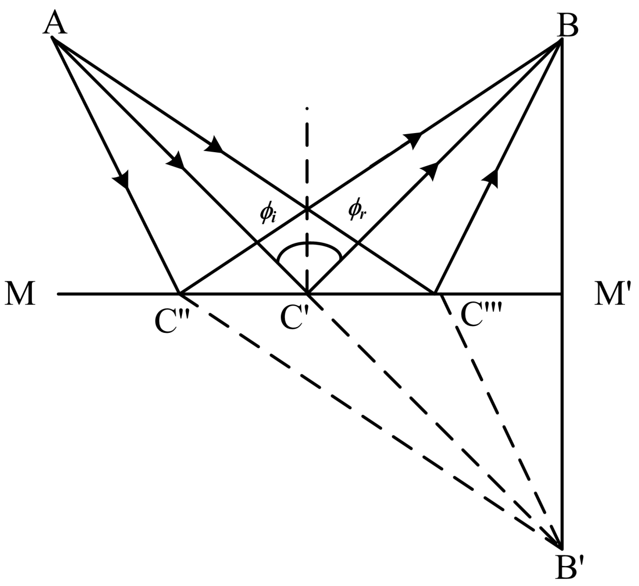

Figure 1.17. Illustration of Fermat's principle.

conditions. Instead, let us use *Fermat's principle* to prove it. There are infinite number of paths to go from point A to point B after striking the mirror. Fermat's principle can be loosely stated as follows: out of infinite number of paths to go from point A to point B, light chooses the path that takes the shortest transit time. In Fig. 1.17, light could choose AC$'$B, AC$''$B, AC$'''$B or any other path. But it chooses the path AC$'$B for which $\phi_i = \phi_r$. Draw the line M$'$B$'$ = BM$'$ so that BC$'$ = C$'$B$'$, BC$''$ = C$''$B$'$ and so on. If AC$'$B$'$ is a straight line, it would be the shortest of all the paths connecting A and B$'$. Since AC$'$B(= AC$'$B$'$), it would be the shortest path to go from A to B after striking the mirror and therefore, according to Fermat's principle, light chooses the path AC$'$B which takes the shortest time. To prove that $\phi_i = \phi_r$, consider the point C$'$. Adding up all the angles at C$'$, we find

$$\phi_i + \phi_r + 2(\pi/2 - \phi_r) = 2\pi \tag{1.138}$$

or

$$\phi_i = \phi_r \tag{1.139}$$

1.9.1 Refraction

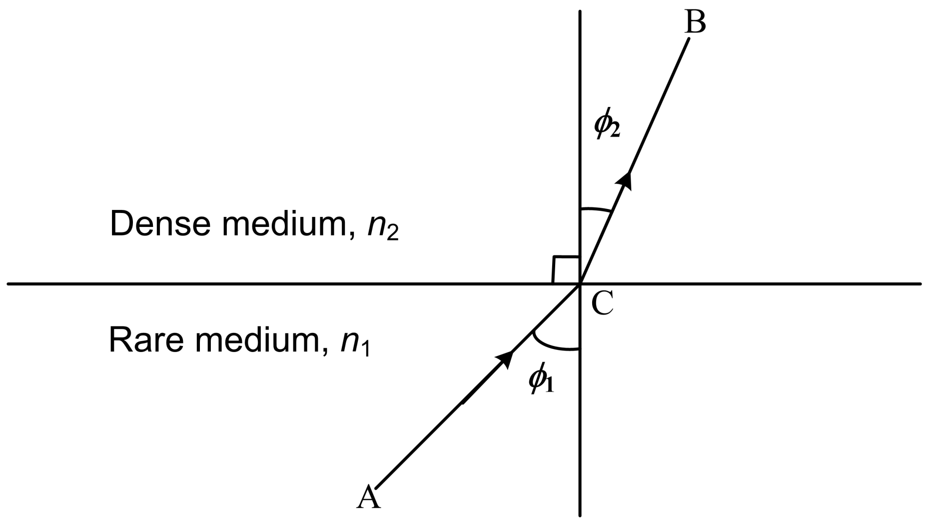

Figure 1.18. Refraction of a plane wave incident at the interface of two dielectrics.

In a medium with constant refraction index, light travels in a straight line. But as the light travels from rarer medium to denser medium, it bends towards the normal to the interface as shown in Fig. 1.18. This phenomenon is called *refraction*, and it can be explained using Fermat's principle. Since the speed of light in two media are different, the path which takes the shortest time to reach B from A may not be a straight line AB. Feynmann *et al* [2] give the following analogy: suppose there is a little girl drowning in a sea at point B and screaming for help as illustrated in Fig. 1.19. You are at point A on the land. Obviously the paths AC$_2$B and AC$_3$B take longer time. You could choose

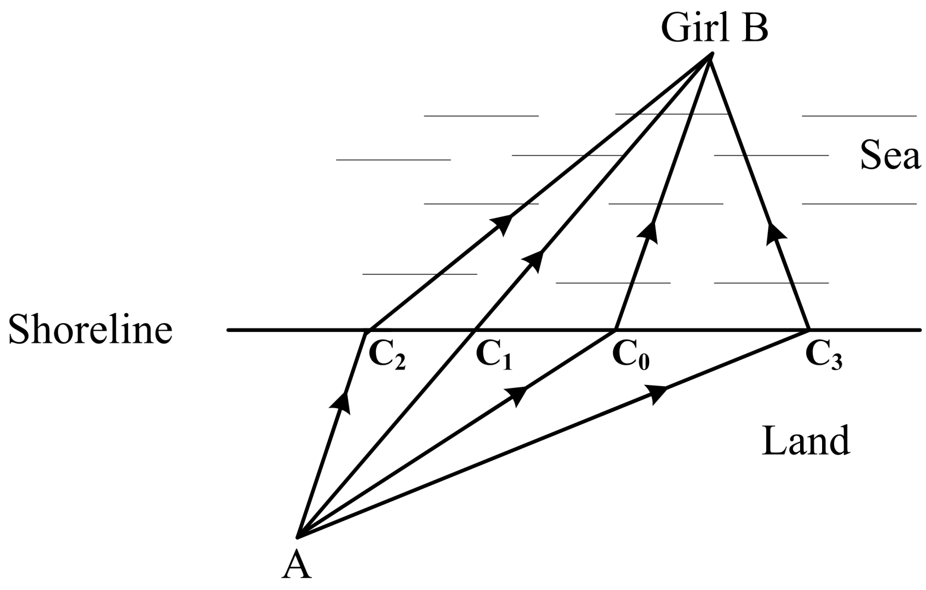

Figure 1.19. Different paths to connect A and B

the straight line path AC_1B. But since running takes less time than swimming, it is advantageous to travel a little longer distance on the land than on the sea. Therefore, the path AC_0B would take shorter time than AC_1B. Similarly, in the case of light propagating from rare medium to dense medium (Fig. 1.20), light travels faster in the rare medium and therefore, the path AC_0B may take shorter time than AC_1B. This explains why light bends towards the normal. To obtain a relation between the angle of incidence ϕ_1 and angle of refraction ϕ_2, let us consider the time taken by light to go from A to B via several paths.

$$t_x = \frac{n_1 AC_x}{c} + \frac{n_2 C_x B}{c}, \quad x = 0, 1, 2, \ldots \tag{1.140}$$

From Fig. 1.21, we have

$$AD = x,\ C_xD = y,\ AC_x = \sqrt{x^2 + y^2} \tag{1.141}$$

$$BE = AF - x,\ BC_x = \sqrt{(AF - x)^2 + BG^2} \tag{1.142}$$

Substituting this in Eq. (1.140), we find

$$t_x = \frac{n_1\sqrt{x^2 + y^2}}{c} + \frac{n_2\sqrt{(AF - x)^2 + BG^2}}{c} \tag{1.143}$$

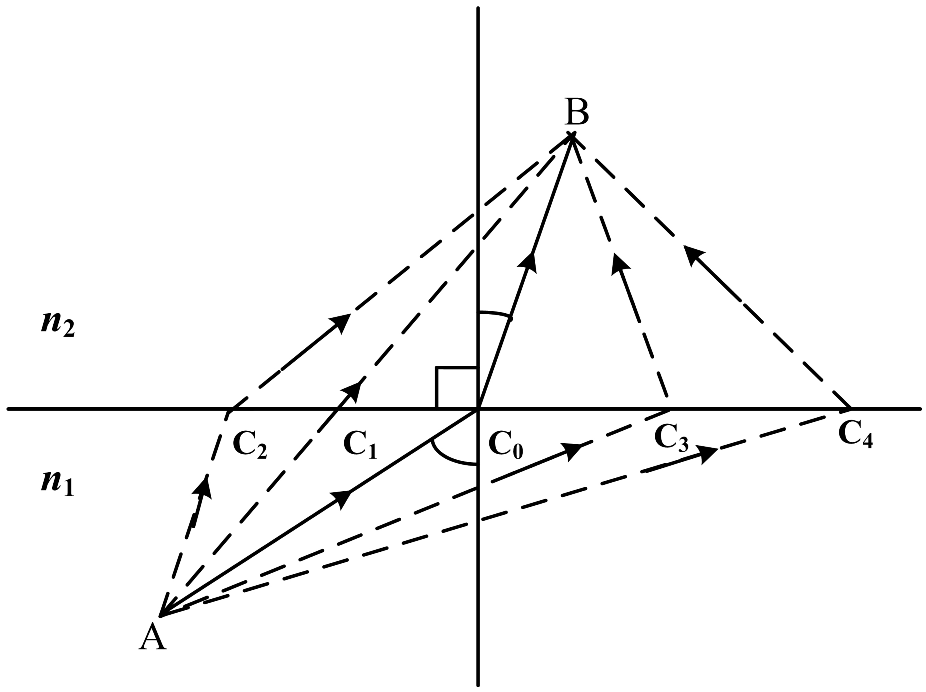

Figure 1.20. Illustration of Fermat's principle for the case of refraction.

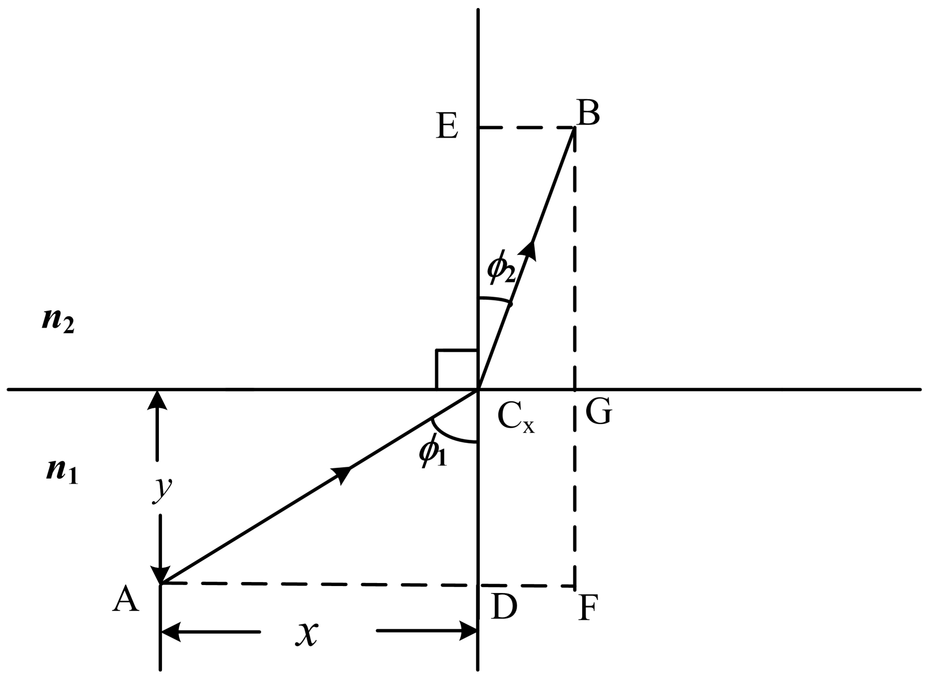

Figure 1.21. Refraction of a lightwave

Note that AF, BG and y are constants as x changes. Therefore, to find the path that takes the least time, we differentiate t_x with respect to x and set it to zero,

$$\frac{dt_x}{dx} = \frac{n_1 x}{\sqrt{x^2+y^2}} - \frac{n_2 (AF - x)}{\sqrt{(AF-x)^2 + BG^2}} = 0 \tag{1.144}$$

From Fig. 1.21, we have

$$\frac{x}{\sqrt{x^2+y^2}} = \sin\phi_1, \quad \frac{AF-x}{\sqrt{(AF-x)^2+BG^2}} = \sin\phi_2 \tag{1.145}$$

Therefore, Eq. (1.144) becomes

$$n_1 \sin\phi_1 = n_2 \sin\phi_2 \tag{1.146}$$

This is called *Snell's law.* If $n_2 > n_1$, $\sin\phi_1 > \sin\phi_2$ and $\phi_1 > \phi_2$. This explains why light bends towards the normal in a denser medium as shown in Fig. 1.18.

When $n_1 > n_2$, from Eq.(1.146), we have $\phi_2 > \phi_1$. As the angle of incidence ϕ_1 increases, the angle of refraction ϕ_2 increases too. For a particular angle, $\phi_1 = \phi_c$, ϕ_2 becomes $\pi/2$,

$$n_1 \sin\phi_c = n_2 \sin\pi/2, \tag{1.147}$$

or

$$\sin\phi_c = n_2/n_1. \tag{1.148}$$

The angle ϕ_c is called the *critical angle.* If the angle of incidence is increased beyond the critical angle, the incident optical ray is completely reflected as shown in Fig.1.22. This is called *total internal reflection* (TIR) and it plays an important role in the propagation of light in optical fibers.

Note that the statement that light chooses the path that takes the least time is not strictly correct. In Fig. 1.16, the time to go from A to B directly (without passing through the mirror) is the shortest and one may wonder why should light go through the mirror. However, if we put constraint that light has to pass through the mirror, the shortest path

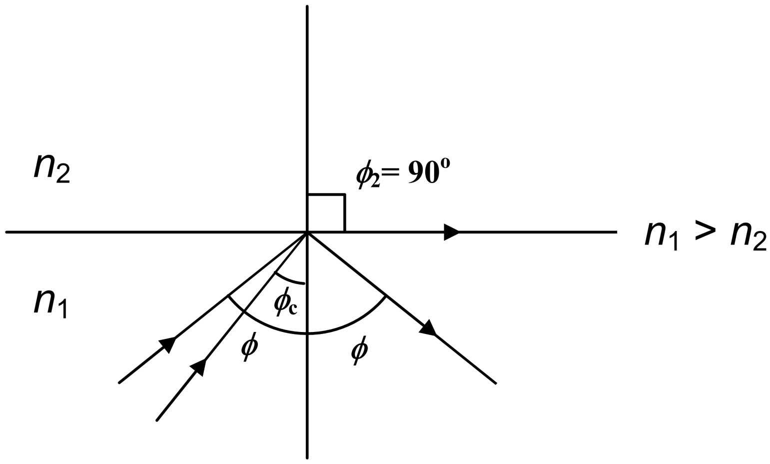

Figure 1.22. Total internal reflection when $\phi > \phi_c$.

would be ACB and light indeed takes that path. In reality, light takes the direct path AB as well as ACB. A more precise statement of Fermat's principle is that light chooses a path for which the transit time is an *extremum*. In fact, there could be several paths satisfying the condition of extremum and light chooses all those paths. By extremum, we mean there could be many neighboring paths and change of time of flight with a small change in the path length is zero to the first order.

Problem 1.2

The critical angle for the glass-air interface is 0.7297 rad. Find the refractive index of glass.

Solution:

Refractive index of air is close to unity. From Eq. (1.148), we have

$$\sin\phi_c = n_2/n_1 \tag{1.149}$$

With $n_2 = 1$, the refractive index of glass, n_2 is

$$\begin{aligned} n_1 &= 1/\sin\phi_c \\ &= 1.5 \end{aligned} \tag{1.150}$$

Problem 1.3

Output of a laser operating at 190 THz is incident on a dielectric medium of refractive index 1.45. Calculate (a) speed of light (b) wavelength in the medium (c) wave number in the medium.

Solution:

(a) Speed of light in the medium is given by

$$v = \frac{c}{n} \tag{1.151}$$

$c = 3 \times 10^8$ m/s, $n = 1.45$,

$$v = \frac{3 \times 10^8 \text{ m/s}}{1.45} = 2.069 \times 10^8 \text{ m/s.} \tag{1.152}$$

(b)

$$\begin{aligned} \text{speed} &= \text{frequency} \times \text{wavelength} \\ v &= f\lambda_m \end{aligned} \tag{1.153}$$

$f = 190$ THz, $v = 2.069 \times 10^8$ m/s,

$$\lambda_m = \frac{2.069 \times 10^8}{190 \times 10^{12}} \text{ m} = 1.0889\ \mu\text{m.} \tag{1.154}$$

(c) Wavenumber in the medium is

$$k = \frac{2\pi}{\lambda_m} = \frac{2\pi}{1.0889 \times 10^{-6}} = 5.77 \times 10^6 \text{ m}^{-1}. \tag{1.155}$$

Problem 1.4

The output of the laser of Problem 1.3 is incident on a dielectric slab with an angle of incidence $= \pi/3$, as shown in Fig. 1.23 (a) Calculate the magnitude of the wave vector

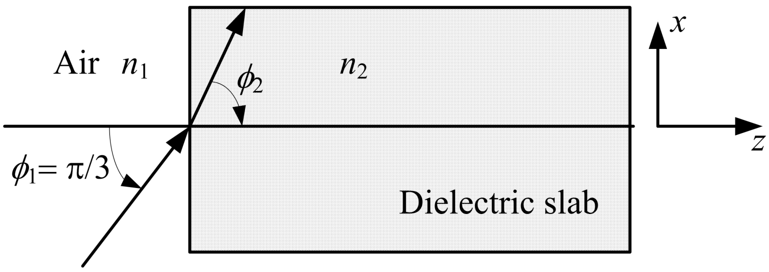

Figure 1.23. Reflection of light at air-dielectric interface.

of the refracted wave (b) calculate the x-component and z-component of the wave vector. The other parameters are the same as in Problem 1.3.

Solution:

Using Snell's law, we have

$$n_1 \sin \phi_1 = n_2 \sin \phi_2, \tag{1.156}$$

For air $n_1 \approx 1$, for the slab $n_2 = 1.45$, $\phi_1 = \pi/3$. So,

$$\phi_2 = \sin^{-1} \left\{ \frac{\sin(\pi/3)}{1.45} \right\} = 0.6401 \text{ rad.} \tag{1.157}$$

The electric field intensity in the dielectric medium can be written as

$$E_y = A \cos(\omega t - k_x x - k_z z) . \tag{1.158}$$

(a) Magnitude of wave vector is same as the wave number, k. It is given by

$$|\mathbf{k}| = k = \frac{2\pi}{\lambda_m} = 5.77 \times 10^6 \text{ m}^{-1}. \tag{1.159}$$

(b) z-component of the wave vector is

$$k_z = k\cos(\phi_2) = 5.77 \times 10^6 \times \cos(0.6401) \ \text{m}^{-1} = 4.62 \times 10^6 \ \text{m}^{-1}. \tag{1.160}$$

x-component of the wave vector is

$$k_x = k\sin(\phi_2) = 5.77 \times 10^6 \times \sin(0.6401) \ \text{m}^{-1} = 3.44 \times 10^6 \ \text{m}^{-1}. \tag{1.161}$$

1.10 Phase Velocity and Group Velocity

Consider the superposition of two monochromatic electromagnetic waves of frequencies $\omega_0 + \Delta\omega/2$ and $\omega_0 - \Delta\omega/2$ as shown in Fig. 1.24. Let $\Delta\omega \ll \omega_0$. The total electric field

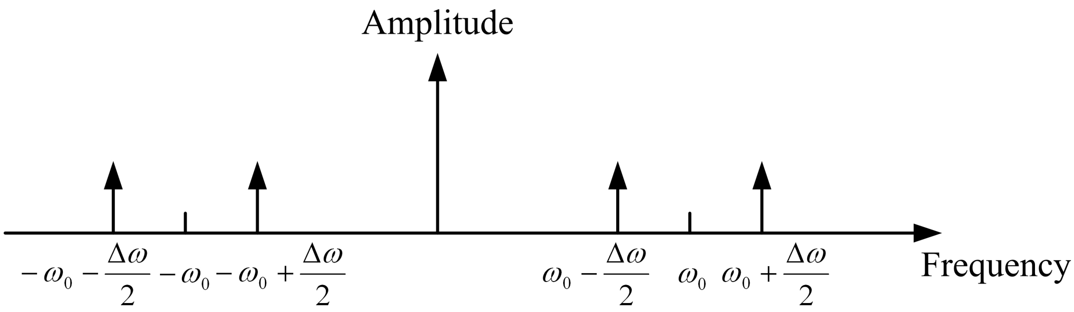

Figure 1.24. The spectrum when two monochromatic waves are superposed.

intensity can be written as

$$E = E_1 + E_2 \tag{1.162}$$

Let the electric field intensity of these waves be

$$E_1 = \cos[(\omega_0 - \Delta\omega/2)t - (k - \Delta k/2)z] \tag{1.163}$$

$$E_2 = \cos[(\omega_0 + \Delta\omega/2)t - (k + \Delta k/2)z] \tag{1.164}$$

Using the formula,

$$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

Eq. (1.162) can be written as

$$E = 2\underbrace{\cos(\Delta\omega t - \Delta k z)}_{\text{field envelope}}\underbrace{\cos(\omega_0 t - k_0 z)}_{\text{carrier}} \tag{1.165}$$

Eq. (1.165) represents the modulation of an optical carrier of frequency ω_0 by a sinusoid of frequency $\Delta\omega$. Fig. 1.25 shows the total electric field intensity at $z = 0$. The broken line shows the field envelope and the solid line shows the rapid oscillations due to optical carrier. We have seen before that

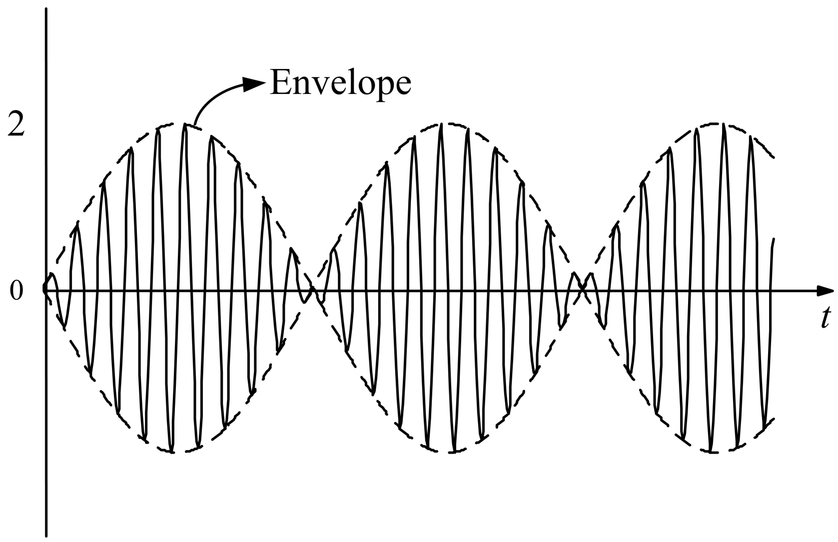

Figure 1.25. Superposition of two monochromatic electromagnetic waves. The broken lines and solid lines show the field envelope and optical carrier, respectively.

$$v_{ph} = \frac{\omega_0}{k_0}$$

is the velocity of the carrier. It is called the *phase velocity.* Similarly, from Eq.(1.165), the speed with which the envelope moves is given by

$$v_g = \frac{\Delta\omega}{\Delta k} \tag{1.166}$$

where v_g is called the *group velocity.* Even if the number of monochromatic waves traveling together is more than two, an equation similar to Eq.(1.165) can be derived. In general, the speed of the envelope (group velocity) could be different from that of the carrier. However, in free space,

$$v_g = v_{ph} = c$$

The above result can be proved as follows. In free space, the velocity of light is independent of frequency,

$$\frac{\omega_1}{k_1} = \frac{\omega_2}{k_2} = c = v_{ph} \tag{1.167}$$

Let

$$\begin{aligned} \omega_1 &= \omega_0 - \frac{\Delta\omega}{2}, \qquad k_1 = k_0 - \frac{\Delta k}{2} &(1.168)\\ \omega_2 &= \omega_0 + \frac{\Delta\omega}{2}, \qquad k_2 = k_0 + \frac{\Delta k}{2} &(1.169)\end{aligned}$$

From Eqs. (1.168) and (1.169), we obtain

$$\frac{\omega_2 - \omega_1}{k_2 - k_1} = \frac{\Delta\omega}{\Delta k} = v_g \tag{1.170}$$

From Eq.(1.167), we have

$$\omega_1 = ck_1$$

$$\omega_2 = ck_2$$

$$\omega_1 - \omega_2 = c(k_1 - k_2) \tag{1.171}$$

Using Eqs. (1.170) and (1.171), we obtain

$$\frac{\omega_1 - \omega_2}{k_1 - k_2} = c = v_g \tag{1.172}$$

In a dielectric medium, the velocity of light v_{ph} could be different at different frequencies. In general,

$$\frac{\omega_1}{k_1} \neq \frac{\omega_1}{k_2} \tag{1.173}$$

In other words, the phase velocity v_{ph} is a function of frequency,

$$\begin{aligned} v_{ph} &= v_{ph}(\omega) &(1.174)\\ k &= \frac{\omega}{v_{ph}(\omega)} = k(\omega) &(1.175)\end{aligned}$$

In the case of two sinusoidal waves, the group speed is given by Eq. (1.166),

$$v_g = \frac{\Delta\omega}{\Delta k} \tag{1.176}$$

In general, for an arbitrary cluster of waves, the group speed is defined as,

$$v_g = \lim_{\Delta k \to 0} \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk} \tag{1.177}$$

Sometimes it is useful to define the *inverse group speed* β_1 as

$$\beta_1 = \frac{1}{v_g} = \frac{dk}{d\omega} \tag{1.178}$$

β_1 could depend on frequency. If β_1 changes with frequency in a medium, it is called a *dispersive medium*. Optical fiber is an example for a dispersive medium which will be discussed in detail in Chapter 2. If the refractive index changes with frequency, β_1 becomes frequency dependent. Since

$$k(\omega) = \frac{\omega n(\omega)}{c}, \tag{1.179}$$

From Eq. (1.178), it follows that

$$\beta_1(\omega) = \frac{n(\omega)}{c} + \frac{\omega}{c}\frac{dn(\omega)}{d\omega} \tag{1.180}$$

Another example for a dispersive medium is prism in which the refractive index is different for different frequency components. Consider a white light incident on the prism, as shown in Fig. (1.26). Using Snell's law for the air-glass interface at the left, we find

$$\phi_2(\omega) = \sin^{-1}\left(\frac{\sin\phi_1}{n_2(\omega)}\right) \tag{1.181}$$

where $n_2(\omega)$ is the refractive index of the prism. Thus, different frequency components of a white light travel at different angles as shown in Fig. (1.26). Because of the material dispersion of the prism, a white light is spread into rainbow of colors. Next, let us consider the co-propagation of electromagnetic waves of different angular frequencies in a range $[\omega_1\ \omega_2]$ with the mean angular frequency ω_0 as shown in Fig. (1.27).

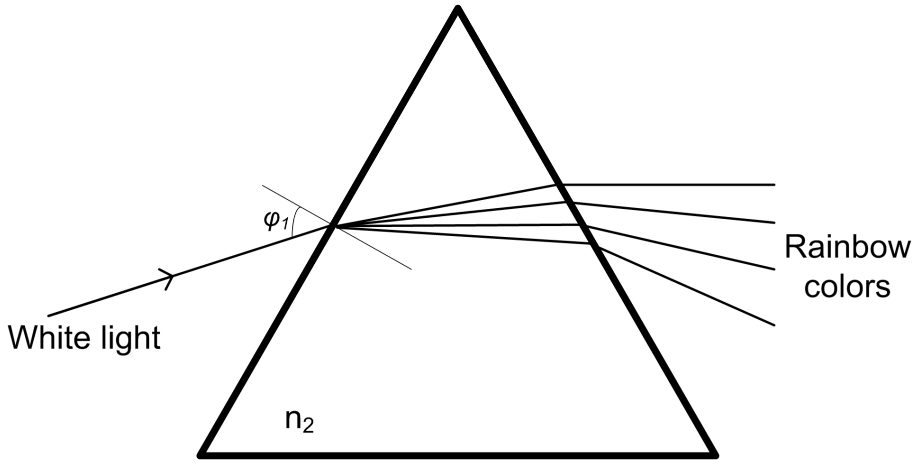

Figure 1.26. Decomposition of white light into its constituter colors.

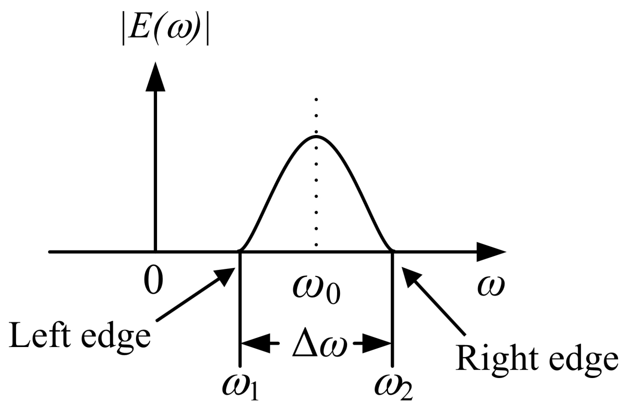

Figure 1.27. The spectrum of electromagnetic wave.

The frequency components near the left edge travel at an inverse speed of $\beta_1(\omega_1)$. If the length of the medium is L, the frequency components corresponding to the left edge would arrive at L after a delay of

$$T_1 = \frac{L}{v_g(\omega_1)} = \beta_1(\omega_1)L$$

Similarly, the frequency components corresponding to the right edge would arrive at L after a delay of

$$T_2 = \beta_1(\omega_2)L$$

The delay between the left edge and right edge frequency components is

$$\Delta T = |T_1 - T_2| = L|\beta_1(\omega_1) - \beta_1(\omega_2)| \tag{1.182}$$

Differentiating Eq. (1.178), we obtain

$$\frac{d\beta_1}{d\omega} = \frac{d^2k}{d\omega^2} \equiv \beta_2 \tag{1.183}$$

β_2 is called the *group velocity dispersion* parameter. When $\beta_2 > 0$, the medium is said to exhibit *normal dispersive.* In the normal-dispersion regime, low-frequency (red-shifted) components travel faster than high-frequency (blue-shifted) components. If $\beta_2 < 0$, the opposite occurs and the medium is said to exhibit *anomalous dispersion.* Any medium with $\beta_2 = 0$ is non-dispersive. Since

$$\frac{d\beta_1}{d\omega} = \lim_{\Delta\omega \to 0} \frac{\beta_1(\omega_1) - \beta_1(\omega_2)}{\omega_1 - \omega_2} = \beta_2, \tag{1.184}$$

and

$$\beta_1(\omega_1) - \beta_1(\omega_2) \simeq \beta_2 \Delta\omega, \tag{1.185}$$

using Eq. (1.185) in Eq. (1.182), we obtain

$$\Delta T = L|\beta_2|\Delta\omega \tag{1.186}$$

In free space, β_1 is independent of frequency, $\beta_2 = 0$ and therefore, the delay between left and right edge components is zero. This means that the pulse duration at the input ($z = 0$) and output ($z = L$) would be the same. However, in a dispersive medium such as optical fiber, the frequency components near ω_1 could arrive earlier (or later) than that near ω_2 leading to pulse broadening.

Problem 1.5

An optical signal of bandwidth 100 GHz is transmitted over a dispersive medium with $\beta_2 = 10$ ps^2/km. The delay between minimum and maximum frequency components is found to be 3.14 ps. Find the length of the medium.

Solution

$$\Delta\omega = 2\pi 100 \text{ Grad/s}, \quad \Delta T = 3.14\text{ps}, \quad \beta_2 = 10 \text{ ps}^2/\text{km}. \tag{1.187}$$

Substituting Eq. (1.187) in Eq. (1.186), we find $L = 500$ m.

1.11 Polarization of Light

So far we have assumed that the electric and magnetic fields of a plane wave are along the x- and y-directions, respectively. In general, electric field can be in any direction in x-y plane. This plane wave propagates in z-direction. The electric field intensity can be written as

$$\mathbf{E} = A_x\mathbf{x} + A_y\mathbf{y}, \tag{1.188}$$

$$A_x = a_x \exp\left[i\left(\omega t - kz\right) + i\phi_x\right], \tag{1.189}$$

$$A_y = a_y \exp\left[i\left(\omega t - kz\right) + i\phi_y\right], \tag{1.190}$$

where a_x and a_y are complex amplitudes of the x- and y-polarization components, respectively, and ϕ_x and ϕ_y are the corresponding phases. Using Eqs. (1.189) and (1.190), Eq. (1.188) can be written as

$$E = \mathbf{a} \exp\left[i\left(\omega t - kz\right) + i\phi_x\right], \tag{1.191}$$

$$\mathbf{a} = a_x\mathbf{x} + a_y \exp\left(i\Delta\phi\right)\mathbf{y}, \tag{1.192}$$

where $\Delta\phi = \phi_y - \phi_x$. Here, $\mathbf{a}$ is the complex field envelope vector. If one of the polarization components vanishes ($a_y = 0$, for example), the light is said to be *linearly polarized* in the direction of the other polarization component (the x direction). If $\Delta\phi = 0$ or π, the light wave is also linearly polarized. This is because the magnitude of $\mathbf{a}$ in this case is $a_x^2 + a_y^2$ and the direction of $\mathbf{a}$ is determined by $\theta = \pm\tan^{-1}(a_y/a_x)$ with respect to x-axis, as shown in Fig. 1.28 The light wave is linearly polarized at an angle θ with respect to

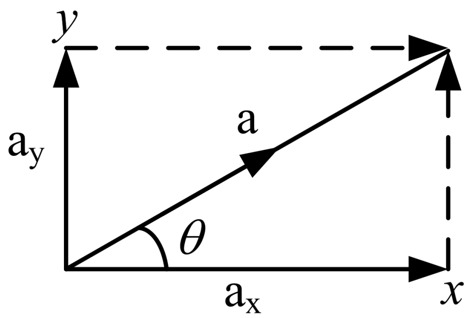

Figure 1.28. The x- and y-polarization components of a plane wave. The magnitude is $|a| = \sqrt{a_x^2 + a_y^2}$ and the angle is $\theta = \tan^{-1}(a_y/a_x)$.

x-axis. A plane wave of angular frequency ω is completely characterized by the complex field envelope vector $\mathbf{a}$. It can also be written in the form of a column matrix known as the *Jones vector*:

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \exp(i\Delta\phi) \end{bmatrix} \tag{1.193}$$

The above form will be used for the description of polarization mode dispersion in optical fibers.

Exercises

1.1 Two identical charges are separated by 1 mm in vacuum. Each of them experience a repulsive force of 0.225 N. Calculate (a) the amount of charge (b) the magnitude of electric field intensity at the location of a charge due to the other charge.
(Ans: (a) 5 nC (b) 4.49 $\times 10^7$ N/C)

1.2 The magnetic field intensity at a distance of 1 mm from a long conductor carrying a

DC current is 239 A/m. The cross-section of the conductor is 2 mm^2. Calculate the (a) current (b)current density.

(Ans: (a) 1.5 A (b) 7.5 $\times 10^5$ A/m^2.)

1.3 Electric field intensity in a conductor due to a time-varying magnetic field is

$$\mathbf{E} = 6\cos(0.1y)\cos(10^5 t)\mathbf{x}\, V/m \tag{1.194}$$

Calculate the magnetic flux density. Assume that the magnetic flux density is zero at t=0.

(Ans: $\mathbf{B} = -0.6\sin(0.1y)\sin(10^6 t)\mathbf{z}\, \mu T$)

1.4 The law of conservation of charges is given by

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

Show that the Ampere's law given by Eq. (1.42) violates the law of conservation of charges and Maxwell's equation given by Eq. (1.43) is in agreement with the law of conservation of charges.

Hint: Take divergence of Eq. (1.42) and use the vector identity

$$\nabla \cdot \nabla \times \mathbf{H} = 0$$

1.5 The x-component of the electric field intensity of the laser operating at 690 nm is

$$E_x(t, 0) = 3\texttt{rect}(t/T_0)\cos(2\pi f_0 t)\, V/m, \tag{1.195}$$

where $T_0 = 5$ ns. The laser and screen are located at $z = 0$ and $z = 5$m, respectively. Sketch the field intensities at the laser and the screen in time and frequency domain.

1.6 Starting from Maxwell's equations (Eqs. (1.48)-(1.51)), prove that the electric field intensity satisfies the wave equation

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

Hint: Take curl on both sides of Eq. (1.50) and use the vector identity

$$\nabla \times \nabla \times \mathbf{E} = \nabla\left(\nabla \cdot \mathbf{E}\right) - \nabla^2 \mathbf{E}$$

1.7 Determine the direction of propagation of the following wave

$$E_x = E_{x0} = \cos\left[\omega\left(t - \frac{\sqrt{3}}{2c}z + \frac{x}{2c}\right)\right]$$

1.8 Show that

$$\Psi = \Psi_0 \exp\left[-\left(\omega t - k_x x - k_y y - k_z z\right)^2\right] \tag{1.196}$$

is a solution of the wave equation (1.125) if $\omega^2 = v^2\left(k_x^2 + k_y^2 + k_z^2\right)$

Hint: Subsititute Eq. (1.196) into the wave equation (1.125)

1.9. A lightwave of wavelength (free space) 600 nm is incident on a dielectric medium of relative permittivity 2.25. Calculate (a) speed of light in the medium (b) frequency in the medium (c) wavelength in the medium (d) wave number in the free space (e) wave number in the medium.

(Ans: (a) $2 \times 10^8 m/s$ (b)500 THz (c) 400 nm (d) $1.047\times10^7 m^{-1}$ (e) $1.57\times10^7 m^{-1}$.)

1.10 State Fermat's principle and explain its applications.

1.11 A light ray propagating in a dielectric medium of index n=3.2 is incident on the dielectric-air interface. Calculate (a) the critical angle (b) if the angle of incidence is $\pi/4$, will it undergo total internal reflection?

(Ans: (a) 0.317 rad. (b) yes)

1.12 Consider a plane wave making an angle of $\pi/6$ radians with the mirror as shown in Fig. 1.29. It undergoes reflection at the mirror and refraction at the glass-air interface. Provide a mathematical expression for the plane wave in the air corresponding to segment CD. Ignore phase-shifts and losses due reflections.

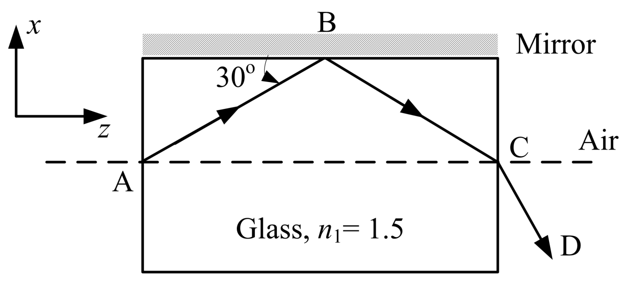

Figure 1.29. Plane wave reflection at the glass-mirror interface.

1.13. Find the average power density of the superposition of N electromagnetic waves given by

$$E_x = \sum_{n=1}^{N} A_n \exp[in(\omega t - kz)] \tag{1.197}$$

1.14 A plane electromagnetic wave of wavelength 400 nm is propagating in a dielectric medium of index n= 1.5. The electric field intensity is

$$\mathbf{E}^{+} = 2\cos(2\pi f_0 t(t - z/v))\mathbf{x}\, V/m. \tag{1.198}$$

(a) Determine the Poynting vector. (b) This wave is reflected by a mirror. Assume that the phase-shift due to reflection is π. Determine the Poynting vector for the reflected wave. Ignore losses due to propagation and mirror reflections.

1.15. An experiment is conducted to calculate the group velocity dispersion coefficient of a medium of length 500 m by sending two plane waves of wavelengths 1550 nm and 1550.1 nm. The delay between these frequency components is found to be 3.92 ps. Find $|\beta_2|$. The transit time for the higher frequency components is found to be less than that for lower frequency component. Is the medium normally dispersive?

(Ans: 100 ps^2/km. No)

Further Reading

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