REVIEW OF DIFFERENTIAL EQUATIONS

CONSIDER THE FIRST ORDER DIFFERENTIAL EQUATION

$$\frac{dy}{dz} = \alpha y \quad \rightarrow (1)$$

where α is CONST. EQ. (1) STATES THAT IF YOU DIFFERENTIATE THE FUNCTION $y(z)$, ITS FUNCTIONAL FORM DOES NOT CHANGE. (IN OTHER WORDS, DERIVATIVE OF THE FUNCTION ~~[illegible]~~ $y(z)$ IS PROPORTIONAL TO $y(z)$). THE ~~ONLY~~ FUNCTION THAT DOES NOT CHANGE ITS FORM ON DIFFERENTIATION IS EXP. FUNCTION.

SO, LET US TRY A TRIAL SOLUTION OF (1) IN THE FORM

$$y(z) = k e^{\lambda z} \quad \rightarrow (2)$$

DIFFERENTIATING (2) w.r.t. z,

$$\frac{dy}{dz} = k\lambda \cancel{e^{\lambda z}} = \alpha y$$

$$= \alpha k \cancel{e^{\lambda z}}$$

$$\therefore \lambda = \alpha;$$

$$y(z) = k e^{\alpha z}$$

AT $z = 0$, $y(0) = k \cdot e^{0} = k$

$$\therefore y(z) = y(0) e^{\alpha z}$$

NEXT, CONSIDER THE SECOND ORDER DIFF. EQ. (2)

$$\frac{d^2y}{dz^2} = \alpha^2 y \quad \text{(3)}$$

α^2 IS A CONST. IT MAY BE REAL OR COMPLEX.

TRY A TRIAL SOLN. IN THE FORM,

$$y = A e^{\lambda z}$$

$$\frac{dy}{dz} = A\lambda e^{\lambda z}$$

$$\frac{d^2y}{dz^2} = \cancel{A}\lambda^2 \cancel{e^{\lambda z}} = \alpha^2 y$$

$$= \alpha^2 \cancel{A} \cancel{e^{\lambda z}}$$

$$\lambda^2 = \alpha^2 \quad \text{(4)}$$

$$\lambda = \pm\alpha \quad \text{(4a)}$$

SO, THERE ARE TWO POSSIBLE SOLUTIONS,

$$y = A_1 e^{\alpha z} \quad \text{(5)}$$

or

$$y = A_2 e^{-\alpha z} \quad \text{(6)}$$

A_1 & A_2 ARE CONSTANTS TO BE DETERMINED BY INITIAL CONDITIONS. SINCE EQ. (3) IS A LINEAR EQUATION, THE GENERAL SOLN. OF (3) IS THE SUPERPOSITION OF (5) & (6), i.e.

$$y = A_1 e^{\alpha z} + A_2 e^{-\alpha z} \longrightarrow (7)$$

CASE (i) α^2 IS -VE. LET

$$\alpha^2 = -\beta^2 \; ; \; \beta^2 > 0.$$

~~FROM~~

FROM (4),

$$\lambda^2 = \alpha^2 = -\beta^2$$

$$\lambda = \pm j\beta$$

$$\therefore y = A_1 e^{j\beta z} + A_2 e^{-j\beta z}$$

CASE (ii) α^2 IS COMPLEX. LET

$$\alpha = \gamma + j\delta \; ; \quad \gamma \text{ \& } \delta \text{ ARE REAL}$$

FROM (4a),

$$\lambda = \pm\alpha = \pm(\gamma + j\delta)$$

$$y = A_1 e^{(\gamma + j\delta) z} + A_2 e^{-(\gamma + j\delta) z}$$

DETERMINATION OF CONSTANTS A_1 & A_2: SUPPOSE $y(0) \equiv y_0$ & $\left.\frac{dy}{dz}\right|_{z=0} \equiv x_0$ ARE KNOWN.

At $z=0$, $y(0) = A_1 e^0 + A_2 e^{-0} = A_1 + A_2 \equiv y_0$

DIFFERENTIATE (7) w.r.t. z,

$$\frac{dy}{dz} = A_1 \alpha . e^{\alpha z} + A_2 (-\alpha) . e^{-\alpha z}$$

$$\left.\frac{dy}{dz}\right|_{z=0} = x_0$$

$\therefore$ At $z=0$,

$$x_0 = A_1 \alpha \cdot e^0 - \alpha A_2 e^{-0}$$

$$x_0 = \alpha \cdot (A_1 - A_2)$$

$$y_0 = A_1 + A_2$$

$$\Rightarrow \quad x_0 + \alpha y_0 = 2\alpha A_1$$

$$\text{or} \quad A_1 = \frac{x_0 + \alpha y_0}{2\alpha}$$

$$\text{lly,} \quad A_2 = \frac{\alpha y_0 - x_0}{2\alpha}$$