SOLUTION OF THE WAVE EQUATION

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 E_x}{\partial t^2} \quad \longrightarrow (1)$$

LET US TRY A TRIAL SOLUTION,

$$E_x = A\cos(\omega t + \beta z) \quad \longrightarrow (2)$$

ω & β ARE CONSTANTS, TO BE DETERMINED.

$$\frac{\partial E_x}{\partial z} = -A\sin(\omega t + \beta z)\cdot\beta$$

$$\frac{\partial^2 E_x}{\partial z^2} = -A\cos(\omega t + \beta z)\,\beta^2 = -\beta^2 E_x \quad \longrightarrow (3)$$

$///^{ly}$, $$\frac{\partial^2 E_x}{\partial t^2} = -\omega^2 E_x \quad \longrightarrow (4)$$

SUBSTITUTING (3) & (4) IN (1), WE FIND

$$-\beta^2 \cancel{E_x} = -\frac{\omega^2}{v^2}\cancel{E_x}$$

$$\beta = \pm\frac{\omega}{v} \quad \longrightarrow (5)$$

CASE (i): $\beta = -\frac{\omega}{v}$

$$E_x = A\cos\left(\omega t - \frac{\omega}{v}z\right) = A\cos\left[\omega\left(t - z/v\right)\right]$$

NOTE: WE ARE UNABLE TO DETERMINE ω. IT HAS TO BE DETERMINED AT THE INITIAL/BOUNDARY CONDITIONS. FOR EXAMPLE, IF THE CURRENT AT THE TRANSMITTER ANTENNA IS AN AC CURRENT @ 1 GHz, THE FREQUENCY OF THE GENERATED EM WAVE IS ALSO 1 GHz & $\omega = 2\pi \times 1 \times 10^9$ RAD/s.

At $t=0$,

$$E_x(z, t=0) = A\cos(-\omega z/v) = A\cos\left(\frac{\omega z}{v}\right) \quad (\because \cos(-\theta) = \cos\theta)$$

At $t = \Delta t$, $\Delta t = +ve$,

$$E_x(z, t=\Delta t) = A\cos(\omega(\Delta t - z/v))$$

$$= A\cos\left(\frac{\omega z}{v} - \omega\Delta t\right)$$

$$= A\cos\left(\frac{\omega z}{v} - \Delta\theta\right) \quad ; \quad \Delta\theta = \omega\Delta t > 0$$

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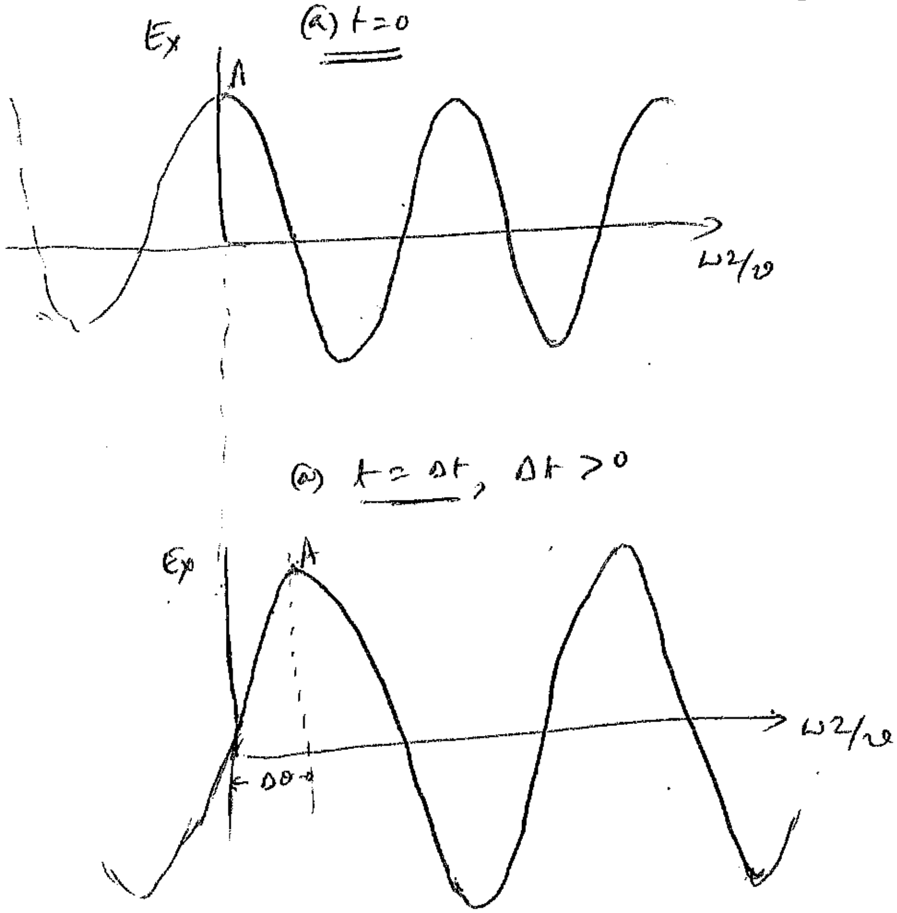


FIG. 1

AS CAN BE SEEN FROM THE FIGURE, AT +VE TIME ($t=\Delta t$), THE WAVE HAS PROPAGATED IN THE +VE Z DIRECTION.

SUCH A WAVE IS KNOWN AS FORWARD PROPAGATING WAVE.

CASE (ii): $\beta = \omega/v$

$$E_x(z,t) = A\cos\left[\omega(t + z/v)\right]$$

At $t=0$,
$$E_x(z, t=0) = A\cos\left(\frac{\omega z}{v}\right)$$

At $t = \Delta t$, $\Delta t = +ve$
$$E_x(z, t=\Delta t) = A\cos\left(\frac{\omega z}{v} + \omega\Delta t\right)$$
$$= A\cos\left(\frac{\omega z}{v} + \Delta\theta\right); \quad \Delta\theta = \omega\Delta t > 0$$
$\to$ (7)

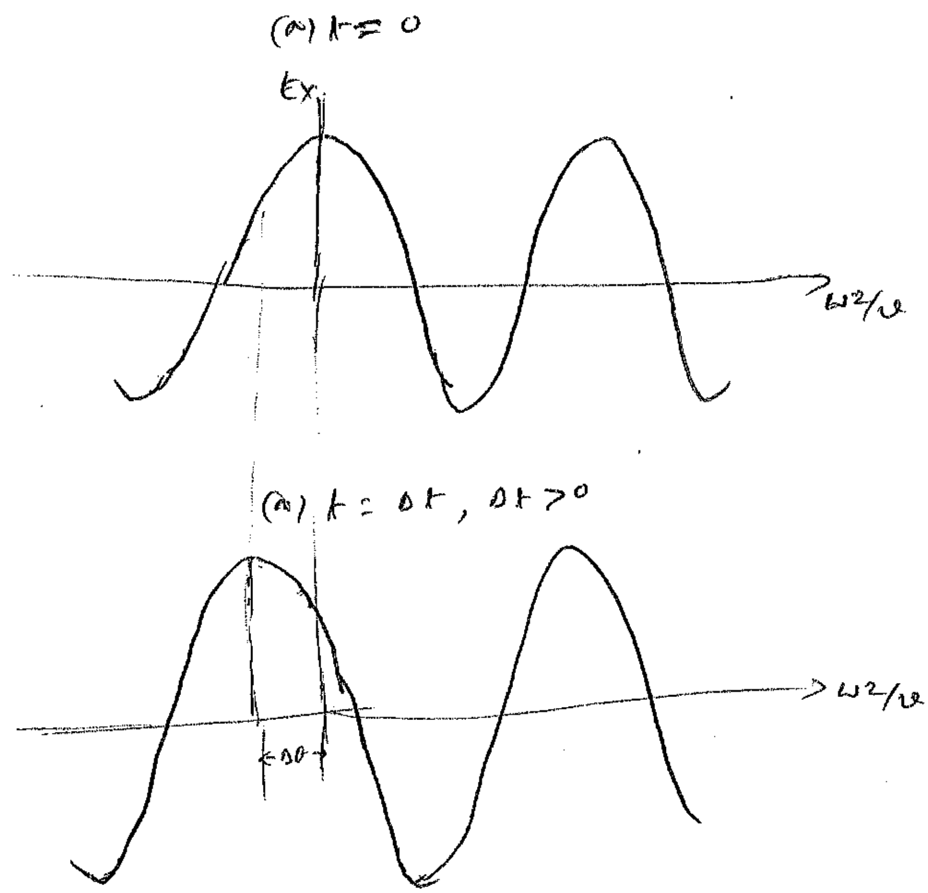


FIG. 2

AS CAN BE SEEN FROM THE FIGURE, AT +VE TIME ($t = \Delta t$), THE WAVE MOVED IN THE -VE Z DIRECTION. THIS WAVE IS CALLED <u>BACKWARD PROPAGATING WAVE</u>.

SUMMARY:

$$E_x(t,z) = A\cos(\omega t - \beta z) \quad : \text{ FORWARD PROP. WAVE}$$

$$E_x(t,z) = A\cos(\omega t + \beta z) \quad : \text{ BACKWARD PROP. WAVE}$$

<u>SPEED, WAVELENGTH & FREQUENCY</u>:

CONSIDER A FORWARD PROPAGATING WAVE SHOWN IN FIG. 1 (p. 2). After THE TIME Δt, THE PEAK POINT A HAS ~~advanced~~ MOVED FORWARD. IN OTHER WORDS, THE PHASE OF THE WAVE HAS CHANGED BY $\Delta\theta = \omega\Delta t$. LOOK AT THE FIG. 1 (BOTTOM): LET $\Delta\theta$ CORRESPOND TO A CHANGE IN DISTANCE Δz, i.e.

$$\Delta\theta = \omega\Delta t = \omega\frac{\Delta z}{v} \quad \text{or}$$

$$\boxed{\frac{\Delta z}{\Delta t} = v} \longrightarrow (8)$$

THE DISTANCE TRAVERSED BY THE WAVE PEAK OVER A TIME INTERVAL ΔT IS $\Delta z = v\Delta t$

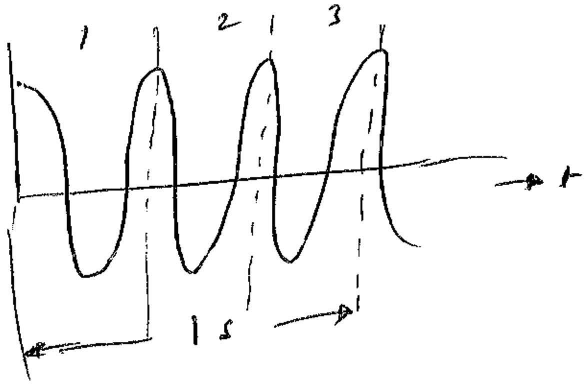
FREQUENCY & PERIOD:

$$E_x(z,t) = A\cos(\omega t - \beta z)$$

At $z=0$, $E_x(0,t) = A\cos(2\pi f t)$

$\omega = 2\pi f$ = ANGULAR FREQ.

EXAMPLE:



SUPPOSE, THE NUMBER OF CYCLES PER SECOND = 3. THIS MEANS $f = 3$ cycles/s $= 3$ Hz.

FROM THE ABOVE FIGURE, WE SEE THAT THE DURATION OF A CYCLE IS $1/3$ s.

SO, DURATION OF A CYCLE = PERIOD $= T = \frac{1}{f} = \frac{1}{3}$ s.

WAVELENGTH & WAVENUMBER:

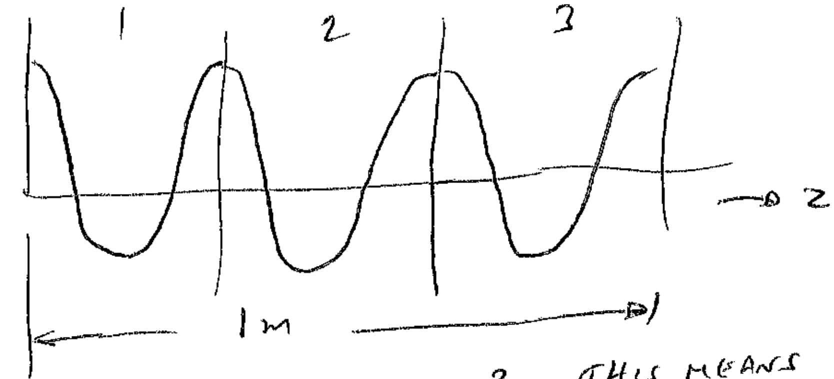
$$E_x(z,t) = A\cos(\omega t - \beta z)$$

At $t=0$, $E_x(z,0) = A\cos(\beta z)$

LET

$$\beta = 2\pi f_s \quad ; \quad f_s = \text{SPATIAL FREQUENCY.}$$

EXAMPLE:



NO. OF CYCLES PER SECOND = 3. THIS MEANS

$$f_s = \cancel{2\ \text{cycles}/s}\ 3\ \text{cycles}/m$$

FROM THE ABOVE FIGURE, WE SEE THAT THE LENGTH OF A CYCLE IS $1/2$ m.

SO, LENGTH OF A CYCLE OR THE PERIOD OF A CYCLE

$$= \lambda = \frac{1}{f_s} = \frac{1}{3}\ m$$

λ IS KNOWN AS THE WAVELENGTH.

$$\therefore \boxed{\beta = 2\pi f_s = \frac{2\pi}{\lambda}} \longrightarrow (9)$$

β IS CALLED THE WAVE NUMBER ~~wh~~ WHICH IS THE SPATIAL ANALOG OF ANGULAR FREQ.

UNIT OF β = rad/m

$$\frac{\omega}{\beta} = v \Rightarrow \frac{2\pi f}{2\pi/\lambda} = v$$

$$\Rightarrow \boxed{f\lambda = v} \longrightarrow (10)$$

PLANE WAVES

TEMPORAL	SPATIAL
FREQUENCY, f	SPATIAL FREQ; f_s
PERIOD, $T = 1/f$	WAVELENGTH, $\lambda = 1/f_s$
ANG. FREQ, $\omega = 2\pi f = \frac{2\pi}{T}$	WAVENUMBER, $\beta = 2\pi f_s = 2\pi/\lambda$

IN FREE SPACE, $\lambda = \lambda_0$; $v = c$

$$\therefore f\lambda_0 = c \longrightarrow (11)$$

AS THE EM WAVE TRAVELS FROM FREESPACE TO A MEDIUM, THE FREQUENCY DOES NOT CHANGE. THE FREQ. IS DETERMINED BY THE SOURCE (ANTENNA CURRENT, FOR EXAMPLE). HOWEVER, THE WAVELENGTH CHANGES.

~~LET~~ IN A MEDIUM, LET $\lambda = \lambda_M$

USING EQ. (10), $f\lambda_M = v \longrightarrow (12)$

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DIVIDING (12) BY (11) & USING

$$v = \frac{c}{n} \quad , \quad n = \text{REFRACTIVE INDEX}$$

WE FIND

$$\boxed{\frac{\lambda_m}{\lambda_0} = \frac{v}{c} = \frac{1}{n}}$$

~~WA~~ SINCE $n > 1$, WAVELENGTH IN A MEDIUM IS ALWAYS LESS THAN THAT IN FREE SPACE.

LET THE WAVE NUMBER IN A MEDIUM BE β_m & THAT IN FREE SPACE BE β_0 — YOU CAN EASILY VERIFY THAT

$$\boxed{\beta_m = \beta_0 n = \frac{2\pi}{\lambda_0} n}$$