Tutorial 2

Problem 1: Find the amplitude of the displacement current density:

(a) adjacent to an automobile antenna where the magnetic field intensity of an FM signal is $H_x = 0.15\cos[3.12(3\times10^8 t - y)]\,\mathrm{A/m}$;

(b) in the air space at a point within a large power distribution transformer where $\mathbf{B} = 0.8\cos[1.257\times10^{-6}(3\times10^8 t - x)]\vec{\mathbf{y}}\,\mathrm{T}$;

(c) within a large, oil-filled power capacitor where $\varepsilon_r = 5$ and $\mathbf{E} = 0.9\cos[1.257\times10^{-6}(3\times10^8 t - z\sqrt{5})]\vec{\mathbf{x}}\,\mathrm{MV/m}$;

(d) in a metallic conductor at 60Hz, if $\varepsilon = \varepsilon_0$, $\mu = \mu_0$, $\sigma = 5.8\times10^7\,\mathrm{S/m}$, and $\mathbf{J} = \sin(377t - 117.1\mathrm{z})\vec{\mathbf{x}}\,\mathrm{MA/m^2}$.

Solutions:

(a)

$$\mathbf{J_d} = \nabla\times\mathbf{H} = -\frac{\partial H_x}{\partial y}\vec{\mathbf{z}} = -0.15\times3.12\sin[3.12(3\times10^8 t - y)]\vec{\mathbf{z}}$$
$$= -0.468\sin[3.12(3\times10^8 t - y)]\vec{\mathbf{z}}\mathrm{A/m^2}$$

So the amplitude is $0.468\mathrm{A/m^2}$.

(b)

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \frac{0.8\cos[1.257\times10^{-6}(3\times10^8 t - x)]\vec{\mathbf{y}}}{4\pi\times10^{-7}}$$
$$= 6.3662\times10^5\cos[1.257\times10^{-6}(3\times10^8 t - x)]\vec{\mathbf{y}}\,\mathrm{A/m}$$

$$\mathbf{J_d} = \nabla\times\mathbf{H} = \frac{\partial H_y}{\partial x}\vec{\mathbf{z}} = 6.3662\times10^5\times1.257\times10^{-6}\sin[1.257\times10^{-6}(3\times10^8 t - x)]\vec{\mathbf{y}}$$
$$= 0.8002\sin[1.257\times10^{-6}(3\times10^8 t - x)]\vec{\mathbf{y}}\,\mathrm{A/m^2}$$

So the amplitude is $0.8002\mathrm{A/m^2}$.

(c)

$$\mathbf{D} = \varepsilon_r\varepsilon_0\mathbf{E} = 5\times8.854\times10^{-12}\times0.9\times10^6\cos[1.257\times10^{-6}(3\times10^8 t - z\sqrt{5})]\vec{\mathbf{x}}$$
$$= 3.9843\times10^{-5}\cos[1.257\times10^{-6}(3\times10^8 t - z\sqrt{5})]\vec{\mathbf{x}}\,\mathrm{C/m^2}$$

$$\mathbf{J}_d = \frac{\partial\mathbf{D}}{\partial t} = -3.9843\times10^{-5}\times1.257\times10^{-6}\times3\times10^8\sin[1.257\times10^{-6}(3\times10^8 t - z\sqrt{5})]\vec{\mathbf{x}}$$
$$= -1.5025\times10^{-2}\sin[1.257\times10^{-6}(3\times10^8 t - z\sqrt{5})]\vec{\mathbf{x}}\,\mathrm{A/m^2}$$

So the amplitude is $0.015025\mathrm{A/m^2}$.

(d)

$$\mathbf{J} = \sigma \mathbf{E} \Rightarrow$$

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} = \frac{10^6 \sin(377t - 117.1z)\vec{\mathbf{x}}}{5.8\times 10^7} = 1.7241\times 10^{-2} \sin(377t - 117.1z)\vec{\mathbf{x}}\ \mathrm{V/m}$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} = 8.854\times 10^{-12} \times 1.7241\times 10^{-2} \sin(377t - 117.1z)\vec{\mathbf{x}}$$
$$= 1.5265\times 10^{-13} \sin(377t - 117.1z)\vec{\mathbf{x}}\ \mathrm{C/m^2}$$

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t} = 1.5265\times 10^{-13} \times 377 \cos(377t - 117.1z)\vec{\mathbf{x}}$$
$$= 5.7549\times 10^{-11} \cos(377t - 117.1z)\vec{\mathbf{x}}\ \mathrm{A/m^2}$$

So the amplitude is 57.549μA/m^2. Here we can see the amplitude of displacement current density $\mathbf{J}_d$, 57.549μA/m^2, is much smaller than the amplitude of the conduction current density $\mathbf{J}$, 10^{6}A/m^2.

Problem 2: Let $\mu = 10^{-5}\ \mathrm{H/m}$, $\varepsilon = 4\times 10^{-9}\ \mathrm{F/m}$, $\sigma = 0$, and $\rho_v = 0$. Find k (including units) so that each of the following pairs of fields satisfies Maxwell's equations:

(a) $\mathbf{D} = 6\vec{\mathbf{x}} - 2y\vec{\mathbf{y}} + 2z\vec{\mathbf{z}}\ \mathrm{nC/m^2}$, $\mathbf{H} = kx\vec{\mathbf{x}} + 10y\vec{\mathbf{y}} - 25z\vec{\mathbf{z}}\ \mathrm{A/m}$;

(b) $\mathbf{E} = (20y - kt)\vec{\mathbf{x}}\ \mathrm{V/m}$, $\mathbf{H} = (y + 2\times 10^6 t)\vec{\mathbf{z}}\ \mathrm{A/m}$

Solutions:

(a)

1) $\nabla\times\mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \Rightarrow$

$$\nabla\times\mathbf{E} = \frac{1}{\varepsilon}\nabla\times\mathbf{D} = \frac{1}{\varepsilon}\left(\frac{\partial D_z}{\partial y} - \frac{\partial D_y}{\partial z}\right)\vec{\mathbf{x}} + \frac{1}{\varepsilon}\left(\frac{\partial D_x}{\partial z} - \frac{\partial D_z}{\partial x}\right)\vec{\mathbf{y}} + \frac{1}{\varepsilon}\left(\frac{\partial D_y}{\partial x} - \frac{\partial D_x}{\partial y}\right)\vec{\mathbf{z}}$$

$$= \frac{10^{-9}}{\varepsilon}\left[\left(\frac{\partial 2z}{\partial y} - \frac{\partial(-2y)}{\partial z}\right)\vec{\mathbf{x}} + \left(\frac{\partial 6}{\partial z} - \frac{\partial 2z}{\partial x}\right)\vec{\mathbf{y}} + \left(\frac{\partial(-2y)}{\partial x} - \frac{\partial 6}{\partial y}\right)\vec{\mathbf{z}}\right] = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} = \mu\frac{\partial(kx\vec{\mathbf{x}} + 10y\vec{\mathbf{y}} - 25z\vec{\mathbf{z}})}{\partial t} = 0$$

2) $\nabla\times\mathbf{H} = \mathbf{J} + \mathbf{J}_d = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t} = \dfrac{\partial \mathbf{D}}{\partial t}$ $(\mathbf{J} = \rho_v \mathbf{v} = \sigma\mathbf{E} = 0) \Rightarrow$

$$\nabla\times\mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\vec{\mathbf{x}} + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\vec{\mathbf{y}} + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\vec{\mathbf{z}}$$

$$= \left(\frac{\partial(-25z)}{\partial y} - \frac{\partial 10y}{\partial z}\right)\vec{\mathbf{x}} + \left(\frac{\partial kx}{\partial z} - \frac{\partial(-25z)}{\partial x}\right)\vec{\mathbf{y}} + \left(\frac{\partial 10y}{\partial x} - \frac{\partial kx}{\partial y}\right)\vec{\mathbf{z}} = 0$$

$$\frac{\partial \mathbf{D}}{\partial t} = 10^{-9}\frac{\partial(6\vec{\mathbf{x}} - 2y\vec{\mathbf{y}} + 2z\vec{\mathbf{z}})}{\partial t} = 0$$

3) $\nabla\cdot\mathbf{B} = 0 \Rightarrow$

$$\nabla \cdot \mathbf{B} = \mu \nabla \cdot \mathbf{H} = \mu(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z}) = \mu(k + 10 - 25) {Wb}/{m^3} = 0 \Rightarrow k = 15 {A}/{m^2}$$

$$(kx \sim {A}/{m} \Rightarrow k \sim {A}/{m^2})$$

4) $\nabla \cdot \mathbf{D} = \rho_v = 0 \Rightarrow$

$$\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0 - 2 + 2 = 0$$

(b)

1) $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \Rightarrow$

$$\nabla \times \mathbf{E} = -\frac{\partial E_x}{\partial y} \vec{\mathbf{z}} = -\frac{\partial (20y - kt)}{\partial y} \vec{\mathbf{z}} = -20 \vec{\mathbf{z}} {V}/{m^2}$$,

$$-\frac{\partial \mathbf{B}}{\partial t} = -\mu \frac{\partial \mathbf{H}}{\partial t} = -\mu \frac{\partial (y + 2 \times 10^6 t)}{\partial t} \vec{\mathbf{z}} = -10^{-5} \times 2 \times 10^6 \vec{\mathbf{z}} = -20 \vec{\mathbf{z}} {V}/{m^2}$$

2) $\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = \frac{\partial \mathbf{D}}{\partial t} \ (\mathbf{J} = \rho_v \mathbf{v} = \sigma \mathbf{E} = 0) \Rightarrow$

$$\nabla \times \mathbf{H} = \frac{\partial H_z}{\partial y} \vec{\mathbf{x}} = 1 \vec{\mathbf{x}} {A}/{m^2}$$

$$\frac{\partial \mathbf{D}}{\partial t} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} = \varepsilon \frac{\partial (20y - kt)}{\partial t} \vec{\mathbf{x}} = -\varepsilon k \vec{\mathbf{x}} = -4 \times 10^{-9} k \vec{\mathbf{x}} {A}/{m^2}$$

$$-4 \times 10^{-9} k = 1 \Rightarrow k = -2.5 \times 10^8 {V}/{ms} \qquad (kt \sim {V}/{m} \Rightarrow k \sim {V}/{ms})$$

3) $\nabla \cdot \mathbf{B} = 0 \Rightarrow \quad \nabla \cdot \mathbf{B} = \mu \nabla \cdot \mathbf{H} = \mu \frac{\partial (y + 2 \times 10^6 t)}{\partial z} = 0$

4) $\nabla \cdot \mathbf{D} = 0 \Rightarrow \quad \nabla \cdot \mathbf{D} = \varepsilon \nabla \cdot \mathbf{E} = \varepsilon \frac{\partial (20y - kt)}{\partial x} = 0$

Problem 3: The parallel-plate transmission line shown in the figure below has dimensions b=4cm and d=8mm, while the medium between the plates is characterized by $\mu_r = 1$, $\varepsilon_r = 20$, and $\sigma = 0$. Neglect fields outside the dielectric. Given the field $\mathbf{H} = 5\cos(10^9 t - \beta z) \vec{\mathbf{y}} {A}/{m}$, use Maxwell's equations to help find

(a) β, if $\beta > 0$;

(b) The displacement current density at z=0;

(c) The total displacement current crossing the surface x=0.5d, 0<y<b,0<z<0.1m in the $\vec{\mathbf{x}}$ direction.

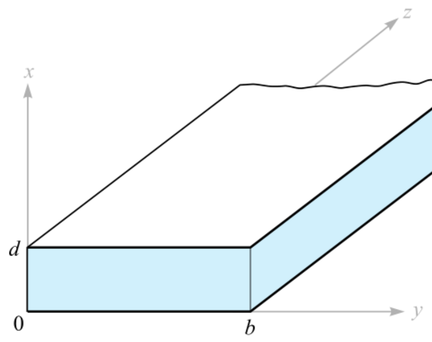

Figure 9.7 See Problem 9.18.

Solutions:

(a) We start with the wave equation: $\frac{\partial^2 H_y}{\partial z^2} = \varepsilon\mu \frac{\partial^2 H_y}{\partial t^2}$

Brief proof:

As $\mathbf{J} = \sigma \mathbf{E} = 0$, so $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$,

$$\nabla \times \mathbf{H} = -\frac{\partial H_y}{\partial z}\vec{\mathbf{x}} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} \Rightarrow \frac{\partial H_y}{\partial z} = -\varepsilon \frac{\partial E_x}{\partial t}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \nabla \times \mathbf{E} = \frac{\partial E_x}{\partial z}\mathbf{y}, -\mu \frac{\partial \mathbf{H}}{\partial t} = -\mu \frac{\partial H_y}{\partial t}\mathbf{y} \Rightarrow \frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t}$$

$$\frac{\partial^2 H_y}{\partial z^2} = \frac{\partial}{\partial z}(-\varepsilon \frac{\partial E_x}{\partial t}) = -\varepsilon \frac{\partial}{\partial t}\frac{\partial E_x}{\partial z} = -\varepsilon \frac{\partial}{\partial t}(-\mu \frac{\partial H_y}{\partial t}) = \varepsilon\mu \frac{\partial^2 H_y}{\partial t^2} \#$$

$$\varepsilon = \varepsilon_r \varepsilon_0 = 20 \times 8.854 \times 10^{-12} = 1.77 \times 10^{-10}\ {}^{F}\!/_{m}$$

$$\mu = \mu_r \mu_0 = 1 \times 4\pi \times 10^{-7} = 4\pi \times 10^{-7}\ {}^{H}\!/_{m}$$

$$\frac{\partial^2 H_y}{\partial z^2} = -5\beta^2 \cos(10^9 t - \beta z)$$

$$\varepsilon\mu \frac{\partial^2 H_y}{\partial t^2} = -1.77 \times 10^{-10} \times 4\pi \times 10^{-7} \times 5 \times 10^{18} \cos(10^9 t - \beta z) = -1.11 \times 10^3 \cos(10^9 t - \beta z)$$

$$\Rightarrow -5\beta^2 \cos(10^9 t - \beta z) = -1.11 \times 10^3 \cos(10^9 t - \beta z) \Rightarrow \beta^2 = 222.54\ {}^{1}\!/_{m^2}$$

$$\beta = 14.92\ {}^{1}\!/_{m} \quad (\beta > 0)$$

(b)

$$\mathbf{H} = 5\cos(10^9 t - 14.92z)\vec{\mathbf{y}}\ {}^{A}\!/_{m}$$

$$\mathbf{J_d} = \nabla \times \mathbf{H} = -\frac{\partial H_y}{\partial z}\vec{\mathbf{x}} = -14.92 \times 5\sin(10^9 t - 14.92z)\vec{\mathbf{x}} = -74.59\sin(10^9 t - 14.92z)\vec{\mathbf{x}}\ \mathrm{A/m^2}$$

$$\mathbf{J_d}(z = 0) = -74.59\sin 10^9 t\vec{\mathbf{x}}\ \mathrm{A/m^2}$$

(c)

$$I = \iint \mathbf{J_d}\vec{\mathbf{x}} dS$$

$$= \int_0^b dy \int_0^{0.1} -74.59\sin(10^9 t - 14.92z)dz = -74.59 \times 0.04 \left.\frac{\cos(10^9 t - 14.92z)}{14.92}\right|_0^{0.1}$$

$$= -0.2[\cos(10^9 t - 1.492) - \cos 10^9 t]A$$