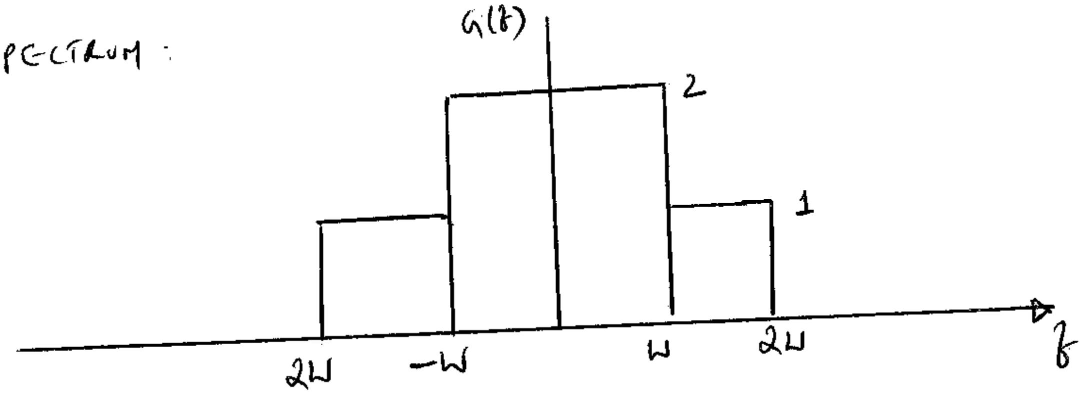
# PRACTICE PROBLEMS

## FOURIER TRANSFORMS

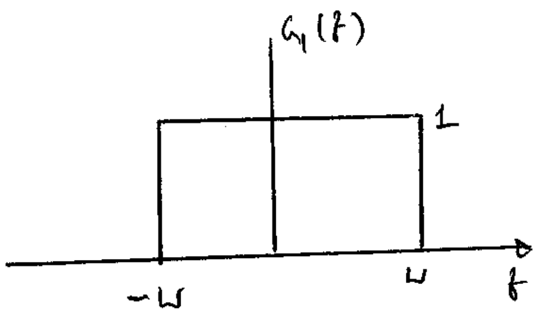
1. FIND THE INVERSE FOURIER TRANSFORM OF THE FOLLOWING SPECTRUM:

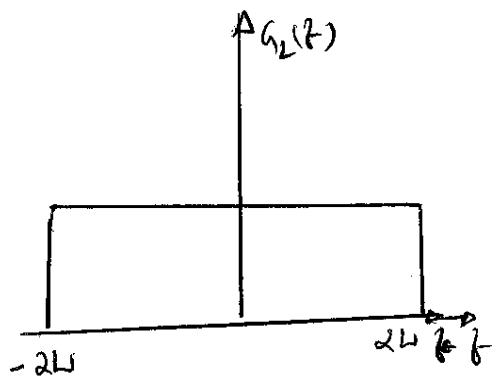



THE SPECTRUM CAN BE DECOMPOSED AS

$$G(f) = G_1(f) + G_2(f)$$

$$G_1(f) = \text{rect}(f/2W), \quad G_2(f) = \text{rect}(f/4W)$$




USE $\text{rect}(f) \rightleftharpoons \text{sinc}(t)$

AND $g(af) \rightleftharpoons \frac{1}{|a|} G(f/a)$, ~~a = 2W~~

WITH $a = 2W$, WE have

$$\mathrm{rect}(f/2W) \rightleftharpoons 2W \cdot \mathrm{sinc}(2Wt)$$

SIMILARLY, $\mathrm{rect}(f/4W) \rightleftharpoons 4W \cdot \mathrm{sinc}(4Wt)$

$\therefore$ THE INVERSE FOURIER TRANSFORM OF $G(f) = g(t)$ IS GIVEN BY

$$g(t) = 2W \cdot \mathrm{sinc}(2Wt) + 4W \cdot \mathrm{sinc}(4Wt)$$

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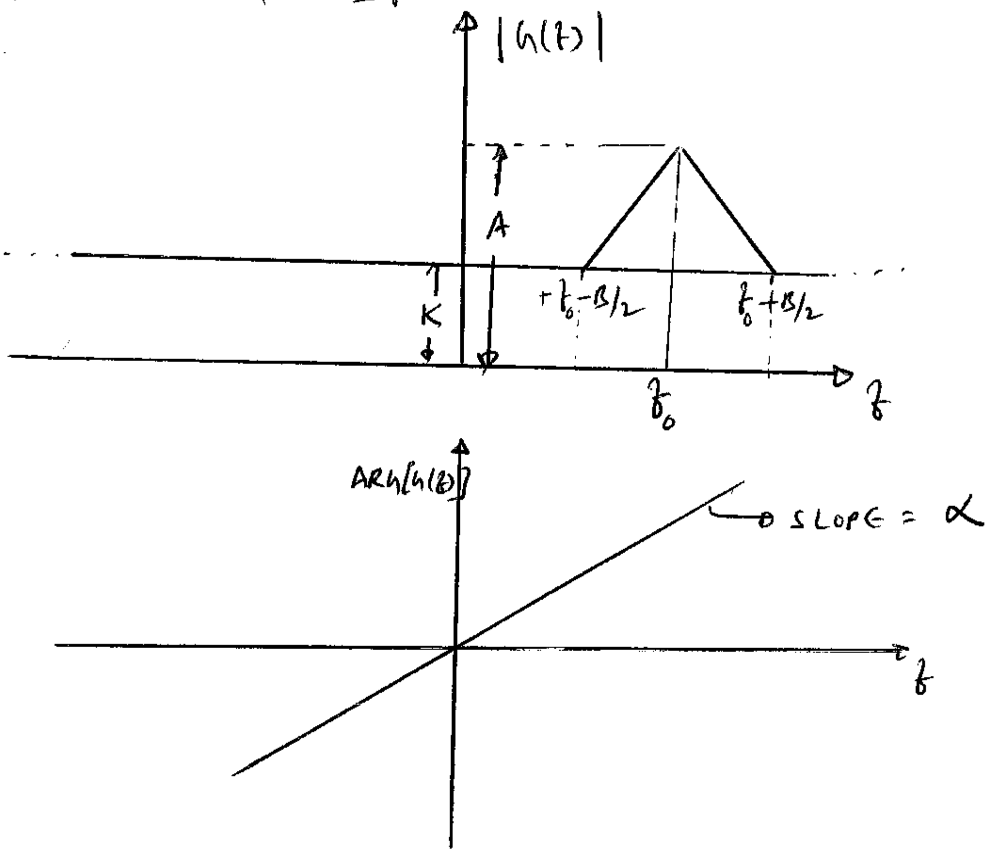
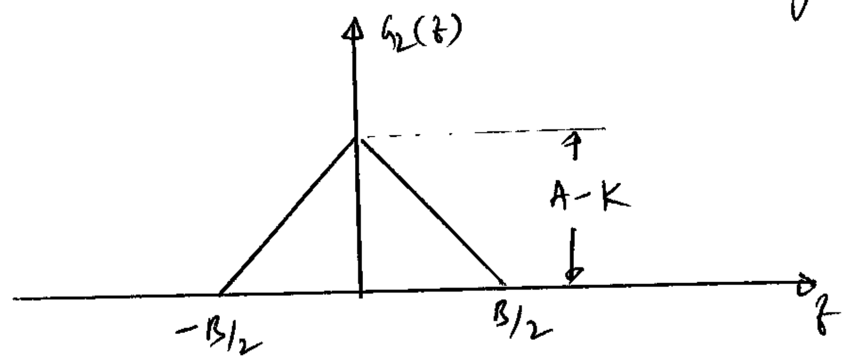
2. FIND THE INVERSE FOURIER TRANSFORM OF THE FOLLOWING SPECTRUM.



Let $G_1(f) = K$ & $G_2(f)$ be the following function:




Let us define a triangular function in the following way:

$$\text{triang}(f) = 1 - |f| \quad \text{for } |f| < 1$$
$$= 0 \quad \text{otherwise}$$

Now $G_2(f)$ can be expressed as

$$G_2(f) = (A-K)\cdot B/2 \cdot \text{triang}\left(f/(B/2)\right)$$

(NOTE: B/2 is half width. A general rule for rect and triang functions: Put the area under the curve (assuming height = 1))

$$\text{rect}(f) * \text{rect}(f) = \text{triang}(f)$$

$$\text{rect}(f) \rightleftharpoons \text{sinc}(t)$$

Using convolution theorem :

$$\text{triang}(f) \rightleftharpoons \text{sinc}^2(t)$$

Using scaling property :

$$\text{triang}\left[f/(B/2)\right] \rightleftharpoons B/2 \cdot \text{sinc}^2(Bt/2)$$

$$\therefore \quad G_2(f) \rightleftharpoons \frac{(A-K)\cdot B^2}{4} \text{sinc}^2(Bt/2)$$

$G(f)$ can be expressed as

$$G(f) = \left[G_1(f) + G_2(f-f_0)\cdot e^{j\alpha f} \quad \longrightarrow (1)$$

Let $G_3(f) = G_1(f) + G_2(f-f_0)$

$$G_2(f-f_0) \rightleftharpoons g_2(t)\cdot \exp(j2\pi t f_0).$$

$$\therefore \quad G_2(f-f_0) \rightleftharpoons \frac{(A-K)B^2}{4}\cdot \text{sinc}^2(Bt/2)\cdot\exp(j2\pi t f_0)$$

$$G_1(f) \rightleftharpoons K\cdot\delta(t)$$

$$\therefore \quad G_3(f) \rightleftharpoons K\,\delta(t) + \frac{(A-K)B^2}{4}\cdot \text{sinc}^2(Bt/2)\,\exp(j2\pi t f_0)$$

From (1)

$$G(f) = G_3(f) \cdot e^{j\alpha f} \longrightarrow (2)$$

Using time shifting property

$$g_3(t-t_0) \rightleftharpoons G_3(f) \cdot e^{-j2\pi f t_0} \longrightarrow (3)$$

Comparing (2) & (3),

$$\alpha = -2\pi t_0$$

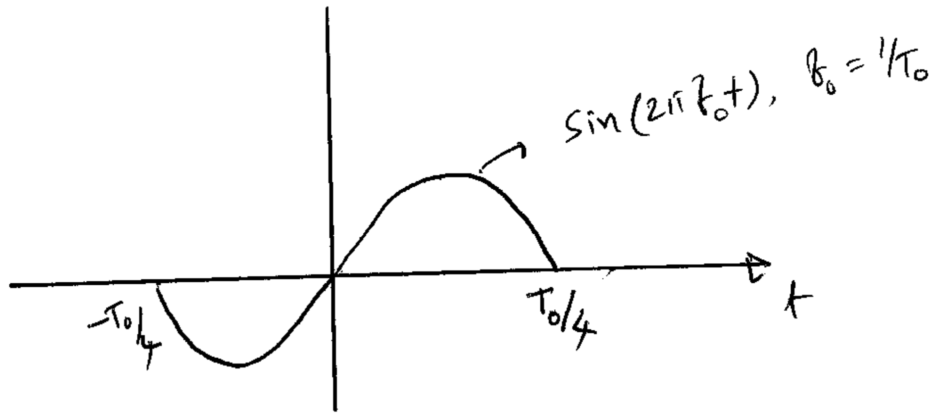
$$\therefore g(t) = F^{-1}[G(f)]$$

$$= g_3(t + \alpha/2\pi)$$

$$= k \cdot \delta\left(t + \frac{\alpha}{2\pi}\right) + \left\{\frac{(A-k)}{4} B^2 \cdot \operatorname{sinc}^2\left[B\left(t + \alpha/2\pi\right)\right] \right.$$

$$\left. * \exp\left[j\, 2\pi t_0 \left(t + \alpha/2\pi\right)\right]\right\}$$

3. Find the Fourier transform of the following 1/2 period sine pulse




Method I: $\cancel{F(g(f))}$

$$F[g(t)] = G(f)$$

$$= \int_{-\infty}^{\infty} g(t) \cdot \exp(-j\, 2\pi f t)\, dt$$

$$= \int_{-T_0/4}^{T_0/2} \sin(2\pi f_0 t) \cdot e^{-j 2\pi f t} \cdot dt$$

$$= \frac{1}{2j} \int_{-T_0/4}^{T_0/4} \left[ e^{j 2\pi f_0 t} - e^{-j 2\pi f_0 t} \right] e^{-j 2\pi f t} \cdot dt$$

$$= \frac{1}{2j} \int_{-T_0/4}^{T_0/4} \left[ e^{j2\pi(f_0 - f)t} - e^{-j2\pi(f_0+f)t} \right] dt$$

$$= \frac{1}{2j} \left\{ \frac{e^{j2\pi(f_0-f)t}}{j2\pi(f_0-f)} + \frac{e^{-j2\pi(f_0+f)t}}{j2\pi(f_0+f} \Bigg|_{-T_0/4}^{T_0/4} \right\}$$

$$= \frac{1}{2j\pi} \left\{ \frac{e^{j2\pi(f_0-f)T_0/4} - e^{j2\pi(f_0-f)\cdot(-T_0/4)}}{j2(f_0-f)} + \frac{e^{-j2\pi(f_0+f)T_0/4} - e^{+j2\pi(f_0+f)T_0/4}}{j2(f_0+f)} \right\}$$

$$= \frac{1}{2j\pi} \left\{ \frac{\sin\left[\frac{\pi T_0}{2}(f_0-f)\right]}{(f_0-f)} - \frac{\sin\left[\frac{\pi T_0}{2}(f_0+f)\right]}{(f_0+f)} \right\}$$

## Method II :

g(t) can be imagined as the product of two functions : (i) Pure sine function, $g_1(t) = \sin(2\pi f_0 t)$ and (ii) A rect function, $g_2(t) = \text{rect}(t/T_0/2)$.

i.e. $g(t) = \sin(2\pi f_0 t) \cdot \text{rect}(t/T_0/2)$

Noting that

$$\sin(2\pi f_0 t) \rightleftharpoons \frac{1}{2j}\left[\delta(f-f_0) - \delta(f+f_0)\right]$$

$$\text{rect}(t/T_0/2) \rightleftharpoons T_0/2 \cdot \text{sinc}(f T_0/2)$$

convolution

$$G(f) \rightleftharpoons \frac{1}{2j}\left[\delta(f-f_0) - \delta(f+f_0)\right] * T_0/2 \cdot \text{sinc}(f T_0/2)$$

$$G(f) \rightleftharpoons \frac{T_0}{4j}\left[\delta(f-f_0) * \text{sinc}(f T_0/2) - \delta(f+f_0) * \sin(f T_0/2\right] \quad (4)$$

Note
$$\delta(f) * X(f) = X(f)$$
$$\delta(f-f_0) * X(f) = X(f-f_0)$$

$$\left[\because \delta(f-f_0) * X(f) = \int_{-\infty}^{\infty} \delta(\lambda - f_0) \cdot x(f-\lambda) \cdot d\lambda = X(f-f_0)\right]$$

$\therefore$ (4) can be written as

$$G(f) \rightleftharpoons \frac{T_0}{4j}\left\{\text{sinc}\left[(f-f_0)\,T_0/2\right] - \text{sinc}\left[(f+f_0)\,T_0/2\right]\right\}$$