Note Title 3/12/2008

SINGLE TONE FREQUENCY MODULATION

LET THE MESSAGE SIGNAL BE

$$m(t) = A_m \cos(2\pi f_m t), \quad A_m > 0$$

THE INSTANTANEOUS FREQUENCY f_i IS

$$f_i = f_c + k_f m(t)$$

$$= f_c + k_f A_m \cos(2\pi f_m t)$$

DEFINE $\Delta f \equiv k_f A_m$. Δf IS KNOWN AS "FREQUENCY DEVIATION"

WHEN $\cos 2\pi f_m t = 1$, $f_i = f_c + \Delta f = f_i^{max}$

WHEN $\cos 2\pi f_m t = -1$, $f_i = f_c - \Delta f = f_i^{min}$

$\therefore$ MAX. FREQUENCY SEPARATION $= f_i^{max} - f_i^{min}$

$= 2\Delta f$

$$\frac{d\varphi}{dt} = 2\pi f_i = 2\pi\left[f_c + \Delta f \cos(2\pi f_m t)\right]$$

Integrating,

$$\varphi(t) = 2\pi f_c t + \frac{\Delta f \sin(2\pi f_m t) \cdot \cancel{2\pi}}{\cancel{2\pi} f_m} + K \longrightarrow (*)$$

WHERE K IS CONST. OF INTEGRATION.

BY CHOOSING $\varphi(0) = 0$ IN $(*)$

$$0 = 0 + 0 + K \Rightarrow K = 0$$

$$\therefore \varphi(t) = 2\pi f_c t + \beta \sin(2\pi f_m t)$$

where $\beta \equiv \dfrac{\Delta f}{f_m} = \dfrac{k_f A_m}{f_m}$ IS KNOWN AS "MODULATION INDEX".

THE FM SIGNAL CAN BE WRITTEN AS

$$s(t) = A_c \cos(\varphi(t))$$

$$\boxed{s(t) = A_c \cos(2\pi f_c t + \beta \sin(2\pi f_m t))}$$

THE ABOVE FORM IS KNOWN AS STANDARD FORM OF SINGLE-TONE FM SIGNAL

FM SPECTRUM:

LET US FIRST FIND THE COMPLEX EXTENSION OF $s(t)$

$$\tilde{s}(t) = A_c e^{j\varphi(t)}$$

$$= A_c e^{j(2\pi f_c t + \beta \sin(2\pi f_m t))}$$

$$\mathrm{Re}[\tilde{s}(t)] = A_c \mathrm{Re}\{e^{j\varphi(t)}\} = A_c \cos(\varphi(t))$$

$$= s(t).$$

$$\tilde{s}(t) = \underbrace{A_c e^{j2\pi f_c t}}_{\text{CARRIER}} \underbrace{e^{j\beta \sin(2\pi f_m t)}}_{\text{complex envelope } m'(t)}$$

$$\tilde{s}(t) = A_c e^{j2\pi f_c t} m'(t)$$

$$m'(t) = e^{j\beta \sin(2\pi f_m t)}$$

HOMEWORK: SHOW THAT $m'(t)$ IS PERIODIC WITH PERIOD $1/f_m$.

HINT: CONSIDER $m'(t+nT_0)$, n integer & show that IT IS EQUAL TO $m'(t)$. $T_0 = 1/f_m$.

SINCE $m'(t)$ IS PERIODIC, IT CAN BE EXPRESSED AS

$$m'(t) = \sum_{n=-\infty}^{\infty} c_n e^{+j2\pi n t f_m}$$

$$c_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} m'(t) e^{-j2\pi n t f_m} dt$$

It can be shown that $c_n = J_n(\beta)$ WHERE $J_n(\beta)$ IS (THIS WILL BE DISCUSSED DURING TUTORIAL) A BESSEL FUNCTION OF nth ORDER.

THERE IS NO CLOSED FORM EXPRESSIONS FOR BESSEL FUNCTIONS.

$J_0(\beta)$ IS THE ZEROTH ORDER BESSEL FUNCTION

$J_1(\beta)$ " " FIRST ORDER " " & SO ON.

TYPICALLY BESSEL FUNCTIONS ARE PLOTTED AS TABLES

$\beta = 2$

n	$J_n(\beta)$
0	x
1	x
2	x
3	x

$\beta = 5$

n	$J_n(\beta)$
0	:
1	:
:	'
:	

FIG 1 SHOWS THE BESSEL FUNCTIONS.

SINCE $\quad c_n = J_n(\beta)$,

$$\tilde{m}(t) = \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi n f_m t}$$

$$\tilde{s}(t) = e^{j2\pi f_c t} \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi n f_m t}$$

$$= \sum_{n=-\infty}^{\infty} J_n(\beta)\, e^{j[2\pi(f_c + n f_m)t]}$$

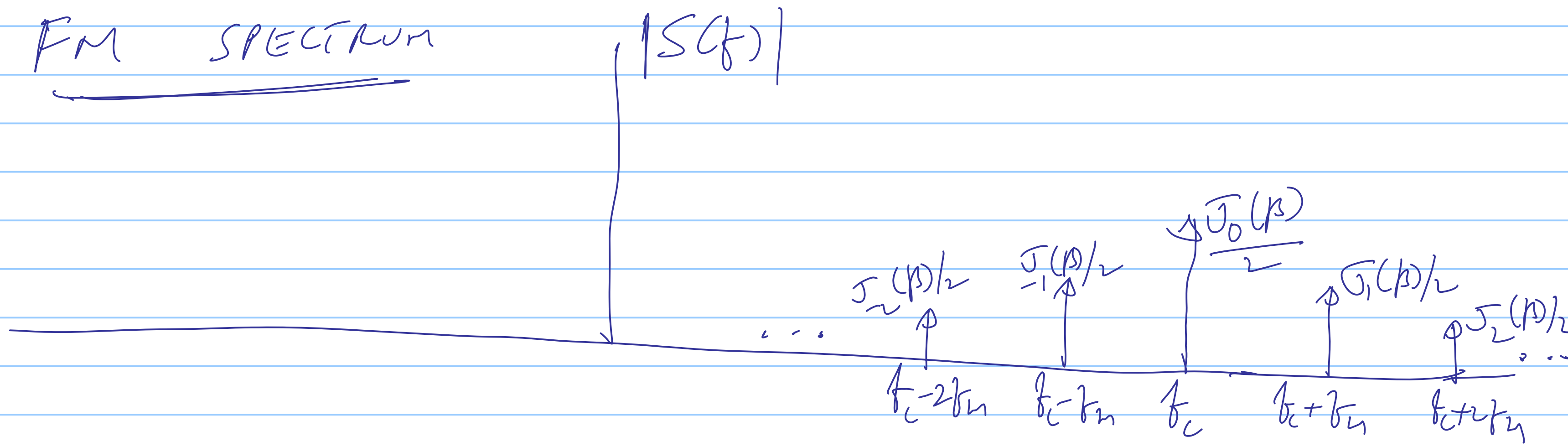
$$s(t) = Re[\tilde{s}(t)] = \sum_{n=-\infty}^{\infty} J_n(\beta)\, Re\left[e^{j[2\pi(f_c + n f_m)t]}\right]$$

$$= \sum_{n=-\infty}^{\infty} J_n(\beta) \cos[2\pi(f_c + n f_m)t]$$

$$\cos(2\pi(f_c + nf_m)t) \rightleftharpoons \frac{1}{2}\left[\delta(f-(f_c+nf_m)) + \delta(f+f_c+nf_m)\right]$$

$$\therefore \quad S(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta)\left[\delta(f-(f_c+nf_m)) + \delta(f+f_c+nf_m)\right]$$

$A_c = 1$ FM SPECTRUM



(ONLY THE SPECTRUM IS DRAWN FOR CONVENIENCE)

FROM THE ABOVE FIGURE WE SEE THAT, FM SPECTRUM HAS COMPONENTS AT $f = f_c, f_c + f_m, f_c - f_m, f_c + 2f_m, f_c - 2f_m$ & SO ON. IN CONTRAST, AM SPECTRUM HAS COMPONENTS AT f_c, $f_c + f_m$ & $f_c - f_m$ ONLY. (FOR SINGLE-TONE MODULATION).

THEREFORE, FM SPECTRUM IS MUCH WIDER THAN AM SPECTRUM, IN GENERAL.

FM CAN BE DIVIDED INTO TWO TYPES - (i) WIDEBAND FM & (ii) NARROWBAND FM.

WIDEBAND FM.

FOR WIDEBAND FM, $\beta \gg 1$. ONE OF THE PROPERTIES OF BESSEL FUNCTIONS IS THAT

WHEN $n > [\beta]$, $J_n(\beta)$ IS NEGLIGIBLY SMALL & HENCE, IT CAN BE IGNORED.

$[\beta]$ MEANS β ROUNDED UP TO THE NEXT-HIGHEST INTEGER

FOR EXAMPLE $\beta = 2.6$, $[\beta] = 3$. IN THIS CASE $J_4(\beta)$, $J_5(\beta)$ & HIGHER ORDER $J_n(\beta)$ CAN BE IGNORED. SIGNIFICANT CONTRIBUTIONS TO THE SPECTRUM COMES FROM COMPONENTS UPTO THIRD HARMONICS.

$\therefore$ BANDWIDTH $= 2 \times 3f_m = 6f_m$

IN GENERAL, BANDWIDTH OF A WIDEBAND FM $= 2[\beta] f_m$

$$\cong 2\beta f_m = 2\frac{\Delta f}{\cancel{f_m}}\cancel{f_m}$$

$$= 2\Delta f$$

NARROW BAND FM ($\beta \ll 1$):

FM SIGNAL CAN BE WRITTEN AS

$$s(t) = A_c \cos\left(2\pi f_c t + \beta \sin(2\pi f_m t)\right)$$

LET $\epsilon = \beta \sin(2\pi f_m t)$

$$s(t) = A_c \cos(2\pi f_c t + \epsilon)$$

$$s(t) = A_c\left[\cos(2\pi f_c t)\cos(\epsilon) - \sin(2\pi f_c t)\sin(\epsilon)\right]$$

WHEN $\beta << 1$, $|\epsilon| << 1$

$$\therefore \sin(\epsilon) \cong \epsilon$$

$$\cos(\epsilon) \cong 1$$

$$s(t) = A_c\left[\cos(2\pi f_c t) - \epsilon \sin(2\pi f_c t)\right]$$

$$= A_c\left[\cos(2\pi f_c t - \beta \sin(2\pi f_m t)\sin(2\pi f_c t)\right]$$

$$2 \sin A \sin B = \cos(A-B) - \cos A \cos B$$

$$\therefore \beta \sin(2\pi f_m t)\sin(2\pi f_c t) = \frac{\beta}{2}\left[\cos(2\pi(f_c - f_m)t) - \cos(2\pi(f_c + f_m)t)\right]$$

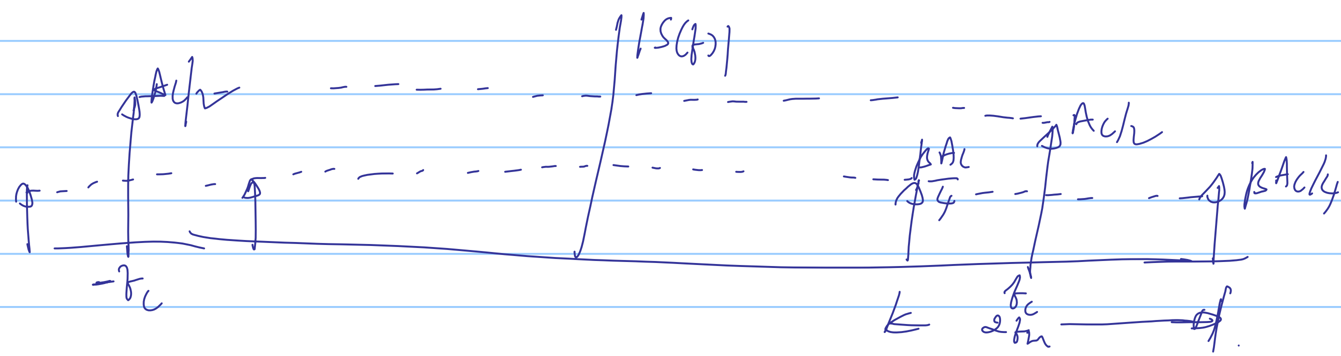
$$A_c \cos(2\pi f_c t) \rightleftharpoons \frac{A_c}{2}\left[\delta(f-f_c)+\delta(f+f_c)\right]$$

$$-\frac{\beta}{2} A_c \cos(2\pi (f_c - f_m) t) \rightleftharpoons -\frac{\beta A_c}{4}\left[\delta(f-(f_c-f_m))+\delta(f+f_c-f_m)\right]$$

$$-\frac{\beta}{2} A_c \cos(2\pi (f_c + f_m) t) \rightleftharpoons -\frac{\beta A_c}{4}\left[\delta(f-(f_c+f_m))+\delta(f+f_c+f_m)\right]$$

COMBINING THE ABOVE EQUATIONS, $S(f)$ CAN BE EVALUATED

THIS FIGURE SHOWS THE MAGNITUDE SPECTRUM



FROM THE FIGURE, WE SEE THAT THE BANDWIDTH OF SINGLE-TONE NARROW BAND FM $= 2f_m$