$$\eta = \frac{\left| \int_{-\infty}^{\infty} H(f)\, X(f) \cdot \exp(j2\pi f T)\, df \right|^2}{N_0/2 \int_{-\infty}^{\infty} |H(f)|^2\, df} \rightarrow (7)$$

NOW WE NEED TO FIND $H(f)$ WHICH MAXIMAZES η. IF $H(f)$ IS AN IDEAL LOW PASS FILTER WITH CUT OFF FREQ. f_c, THE MAXIMUM VALUE OF η CAN BE EASILY OBTAINED BY DIFFERENTIATING EQ.(7) W.R.T. f_c. HOWEVER, WE LIKE TO CONSIDER A MORE GENERAL PROBLEM CORRESPONDING TO $H(f)$ WITH ARBITRARY SHAPE. TO DO THIS JOB, WE MAKE USE OF SCHWARZ'S INEQUALITY.

SCHWARZ'S INEQUALITY:

SUPPOSE WE HAVE TWO COMPLEX FUNCTIONS $\varphi_1(x)$ & $\varphi_2(x)$ IN THE REAL VARIABLE x, THE SCHWARZ'S INEQUALITY STATES THAT

$$\left| \int_{-\infty}^{\infty} \varphi_1(x)\varphi_2(x)\,dx \right|^2 \leq \int_{-\infty}^{\infty} |\varphi_1(x)|^2 dx \int_{-\infty}^{\infty} |\varphi_2(x)|^2 dx \quad \longrightarrow (8)$$

THE EQUALITY HOLDS IF AND ONLY IF

$$\varphi_1(x) = k\,\varphi_2^*(x) \quad \longrightarrow (9)$$

WHERE k IS AN ARBITRARY CONSTANT.

WE NOW USE EQ.( ) TO MAXIMIZE THE PEAK PULSE SNR η.

LET $\varphi_1(x) \triangleq H(f)$ AND $\varphi_2(x) \triangleq X(f)\cdot \exp(j2\pi f T)$.

$$|\varphi_1(x)|^2 \triangleq |H(f)|^2 \quad \longrightarrow (10)$$

$$|\varphi_2(x)|^2 = |X(f)|^2 \cdot 1 \quad \longrightarrow (11)$$

FROM EQ.(8), WE SEE THAT

$$\left| \int_{-\infty}^{\infty} H(f)\, X(f)\, e^{j2\pi f T}\, df \right|^2 \leq \int_{-\infty}^{\infty} |H(f)|^2 df \int_{-\infty}^{\infty} |X(f)|^2 df$$

or

$$\frac{\left| \int_{-\infty}^{\infty} H(f)\, X(f) \cdot e^{j2\pi f T}\, df \right|^2}{N_0/2 \cdot \int_{-\infty}^{\infty} |H(f)|^2 df} \leq \frac{2}{N_0} \int_{-\infty}^{\infty} |X(f)|^2 df \quad \rightarrow (12)$$

i.e. $$\eta \leq \frac{2}{N_0} \int_{-\infty}^{\infty} |X(f)|^2 df \qquad \text{(SEE EQ. (7))} \quad \rightarrow (13)$$

FROM RAYLEIGH ENERGY THEOREM, IT FOLLOWS THAT

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} |x(t)|^2 dt = \text{ENERGY 'E' OF THE PULSE.} \quad \rightarrow (14)$$

THEREFORE, EQ. (13) CAN BE WRITTEN AS

$$\eta \leq \frac{2E}{N_0} \longrightarrow (15)$$

THE PEAK PULSE SNR η CAN NOT EXCEED $\frac{2E}{N_0}$

WHICH IMPLIES

$$\boxed{\eta_{max} = \frac{2E}{N_0}} \longrightarrow (16)$$

EQ. (16) STATES THAT THE MAXIMUM SNR IS THE RATIO OF THE PULSE ENERGY E TO THE PSD OF THE WHITE NOISE AT THE FILTER INPUT. NOTE THAT THE η_{max} DOES NOT DEPEND ON THE SHAPE OF INPUT PULSE. THIS IMPLIES THAT TWO PULSES HAVING DIFFERENT PULSE SHAPES (FOR EXAMPLE, SQUARE & TRIANGULAR) BUT THE SAME ENERGY HAVE THE SAME SNR_{MAX} (ASSUMING THE SAME PSD OF WHITE NOISE)

THE EQUALITY IN EQ. (13) HOLDS IF & ONLY IF

$$H(f) = H_{opt}(f) = k X^*(f) \exp(-j2\pi f T) \longrightarrow (17)$$

THE EQUALITY CORRESPONDS TO MAXIMUM SNR AS GIVEN BY EQ. (16). THEREFORE, THE FILTER TRANSFER FUNCTION GIVEN BY EQ. (17) GIVES THE MAXIMUM VALUE OF SNR.

EQ. (17) IMPLIES THAT THE OPTIMUM FILTER TRANSFER FUNCTION IS THE COMPLEX CONJUGATE OF THE INPUT SIGNAL $X(f)$ EXCEPT FOR THE SCALING FACTOR k & THE DELAY TERM $\exp(-j2\pi f T)$.

NOW LET US LOOK AT THE TIME DOMAIN EQUIVALENT OF EQ. (17)

$$h_{opt}(t) \rightleftharpoons H_{opt}(f)$$

$$\therefore h_{opt}(t) = k \int_{-\infty}^{\infty} X^*(f) \exp(-j2\pi f T) \exp(j2\pi f t)\, df \longrightarrow (18)$$

FOR THE REAL INPUT SIGNAL $x(t)$, WE HAVE

$$X^*(f) = X(-f) \quad \text{(19)}$$

$$\therefore \; h_{opt}(t) = k \int_{-\infty}^{\infty} X(-f) \cdot \exp[j2\pi(-f)(T-t)]\,df \quad \text{(20)}$$

Setting $-f = u$

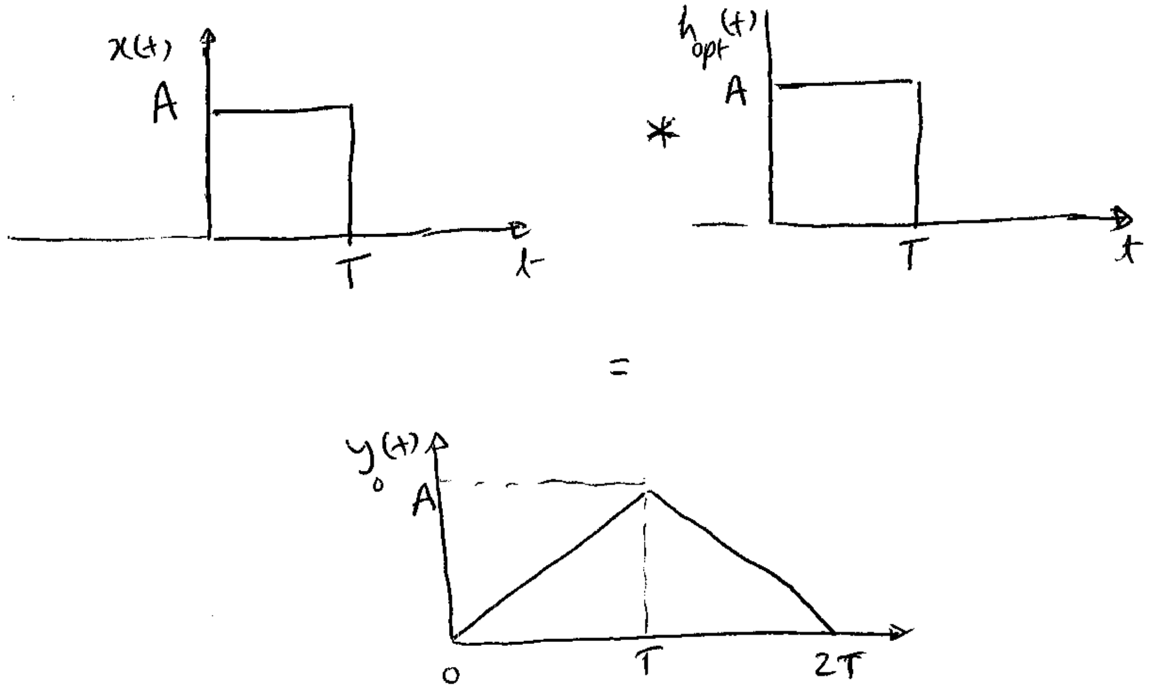
$$h_{opt}(t) = k \int_{-\infty}^{\infty} X(u) \cdot \exp[j2\pi u(T-t)] \cdot du$$

$$\boxed{h_{opt}(t) = k\,x(T-t)} \quad \text{(21)}$$

THE OPTIMUM IMPULSE RESPONSE THAT GIVES THE BEST SNR INTO THE DECISION DEVICE IS SIMPLY A TIME-REVERSED VERSION OF THE SIGNAL PULSE $x(t)$ SHIFTED BY T seconds (BIT PERIOD). THE FILTER $h_{opt}(t)$ GIVEN BY EQ. (21) IS CALLED A <u>MATCHED FILTER</u>. THIS IS DUE TO THE FACT THAT THE OPTIMUM FILTER IS MATCHED TO THE SIGNAL COMPONENT OF THE RECEIVED WAVEFORM.

EXAMPLE: MATCHED FILTER FOR A RECTANGULAR PULSE

LET THE INPUT PULSE $x(t)$ BE A RECTANGULAR PULSE OF DURATION T SECONDS AS SHOWN IN THE FIGURE.



THE OPTIMUM IMPULSE RESPONSE $h_{opt}(t) = x(T-t)$ (ASSUMING $k=1$). AS CAN BE EASILY SEEN, ~~THE~~ $h_{opt}(t)$ IS ALSO A RECTANGULAR PULSE OF DURATION T SECONDS

$$\left(\because\ h_{opt}(0) = x(T) = A,\ h_{opt}(T) = x(0) = A,\ h_{opt}(T+) = x(0-) = 0\right)$$

FOR SIMPLICITY LET US ASSUME $AT = 1$; THE FOURIER TRANSFORM OF $x(t)$ IS GIVEN BY

$$X(f) = \text{sinc}(fT)\exp(-j\pi fT) \quad \text{(22)}$$

THIS IS BECAUSE $x(t)$ IS A RECTANGULAR PULSE DELAYED BY T SECONDS. (USE TIME-SHIFTING PROPERTY TO OBTAIN EQ. (22)). THE MATCHED FILTER $h_{opt}(t)$ IS ~~IDENTICAL~~ EXACTLY SAME AS ~~WITH~~ THE INPUT PULSE $x(t)$ IN THIS EXAMPLE. THEREFORE,

$$H_{opt}(f) = \text{sinc}(fT)\exp(-j\pi fT) \quad \text{(23)}$$

THE SIGNAL COMPONENT OF THE FILTER OUTPUT IS

$$Y_0(f) = X(f)\,H_{opt}(f) = \text{sinc}^2(fT)\exp(-j2\pi fT) \quad \text{(24)}$$

THE TRIANGULAR PULSE DEFINED AS

$$\Delta(t) = 1 - |t| \quad \text{for } |t| < 1 \quad \text{(25)}$$

$$= 0 \quad \text{ELSEWHERE}$$

HAS THE FOURIER TRANSFORM $\text{sinc}^2(f)$, i.e.

$$\Delta(t) \rightleftharpoons \text{sinc}^2(f) \quad \longrightarrow (26)$$

$$A\,\Delta(t/T) \rightleftharpoons AT\,\text{sinc}^2(fT) \quad \text{(SCALING PROPERTY)}$$

$$A\,\Delta\left(\frac{t-T}{T}\right) \rightleftharpoons AT\,\text{sinc}^2(fT)\exp(-j2\pi fT) \quad \longrightarrow (27)$$

THEREFORE, THE SIGNAL COMPONENT OF THE OUTPUT $y_0(t)$ IS GIVEN BY THE TRIANGULAR PULSE DELAYED BY T SECONDS AS SHOWN IN THE FIGURE, i.e.

$$y_0(t) = A\,\Delta\left(\frac{t-T}{T}\right)$$

AS EXPECTED, THE SIGNAL COMPONENT OF THE OUTPUT PULSE $y_0(t)$ HAS THE MAXIMUM VALUE AT $t = T$. THIS IS BECAUSE WE USED THE MATCHED FILTER THAT MAXIMIZES PEAK PULSE SNR AS DEFINED IN EQ. (7).