The Fourier Series

Consider the output of a Function Generator :

$$V_1(t) = 5 \cdot \cos(2\pi \times 5t) \longrightarrow (1)$$

Here the frequency of the signal is 5 Hz. and the peak voltage is 5 volts. As shown below :

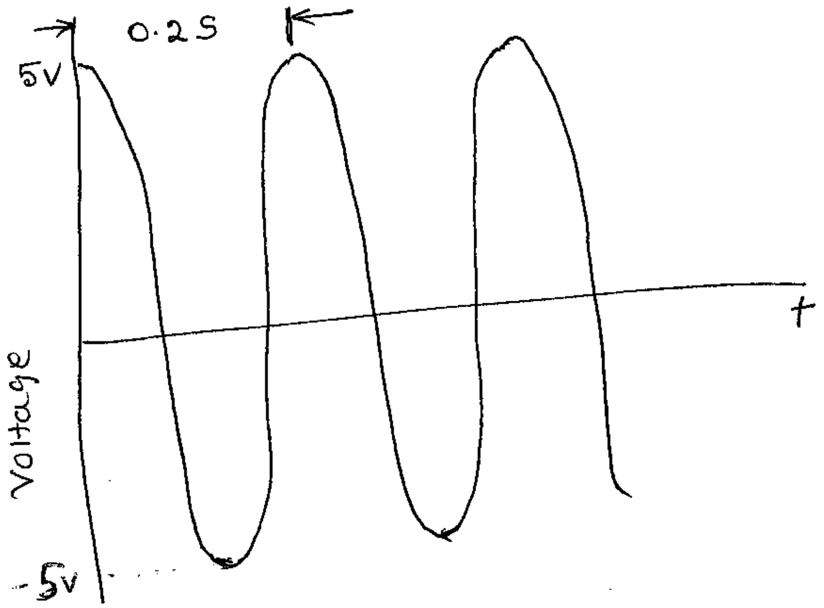

Fig. 1a

The period, T_0 is the inverse of frequency.
In this case, $T_0 = 1/5 = 0.2\,s$.
The output of the signal generator can be represented in frequency domain as

$V_1(t)$

Time Domain

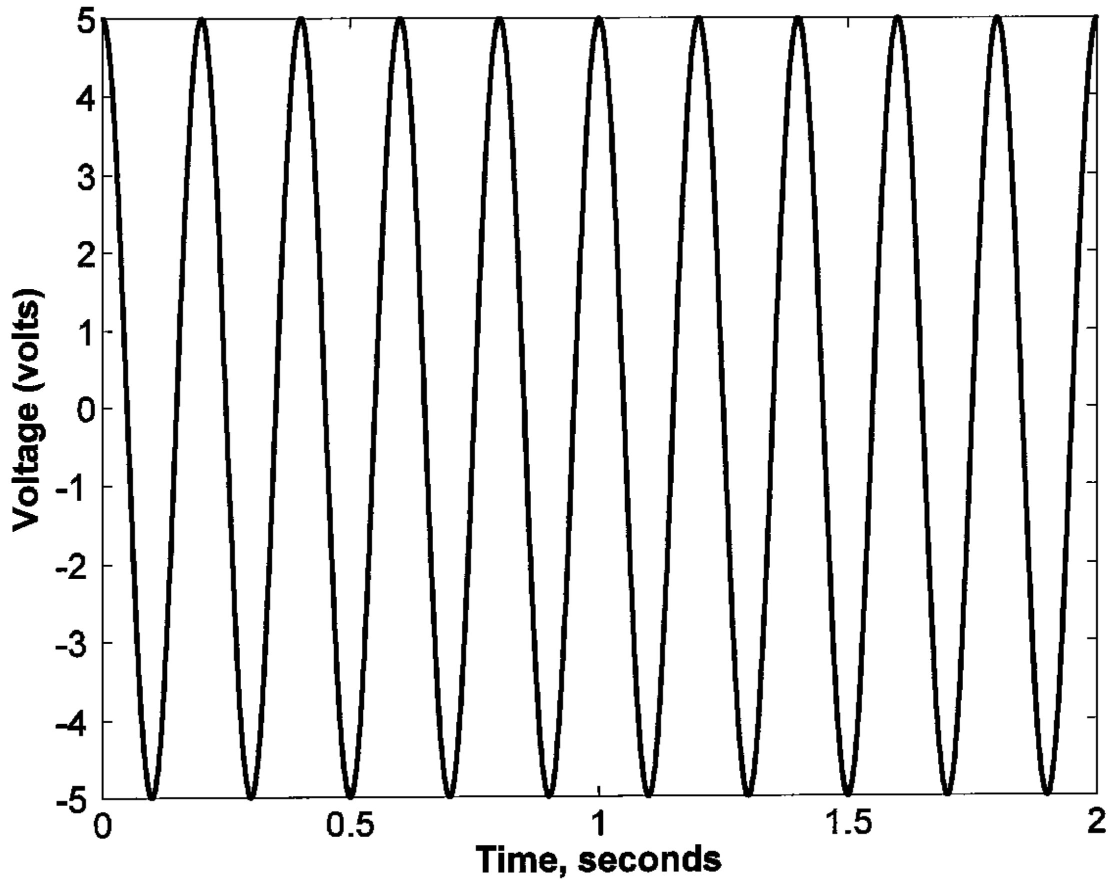

FIG. 1a

Frequency Domain

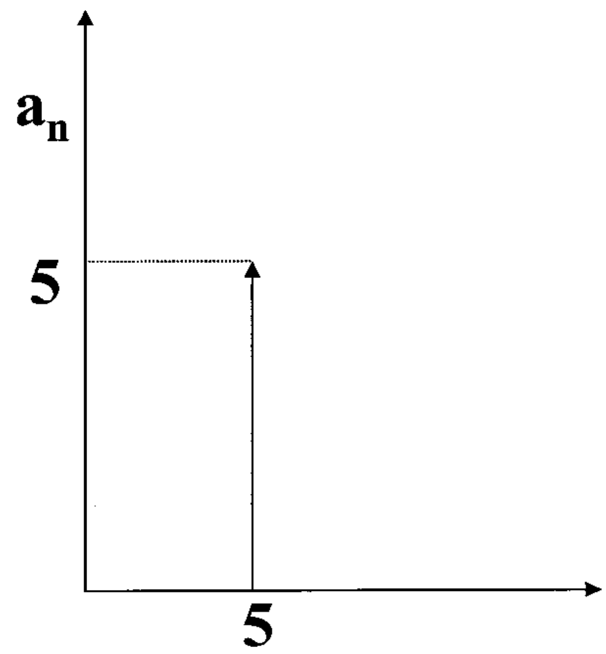

FIG. 1b

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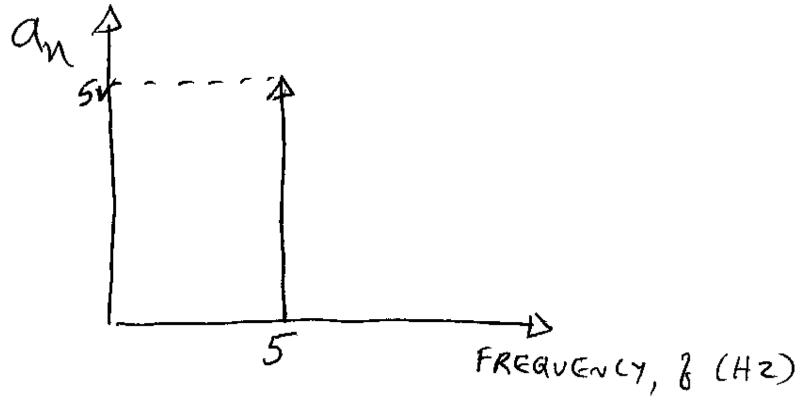

y-axis shows the amplitude of the cosine component.

Let us consider the output of a different function generator given by

$$V_2(t) = 2 \cdot \cos\left[2\pi \times 10 \times (t - 1.11)\right] \longrightarrow (2)$$

The frequency of the above signal generator is 10 Hz and it is delayed by 1.11 seconds with respect to the first signal generator.

Eq. (2) can be written as:

$$V_2(t) = 1.618 \cos(2\pi \times 10t) + 1.1756 \sin(2\pi \times 10t) \longrightarrow (3)$$

Fig.s 2a and 2b show, the signal $V_2(t)$ in time domain and frequency domain, respectively.

Time Domain

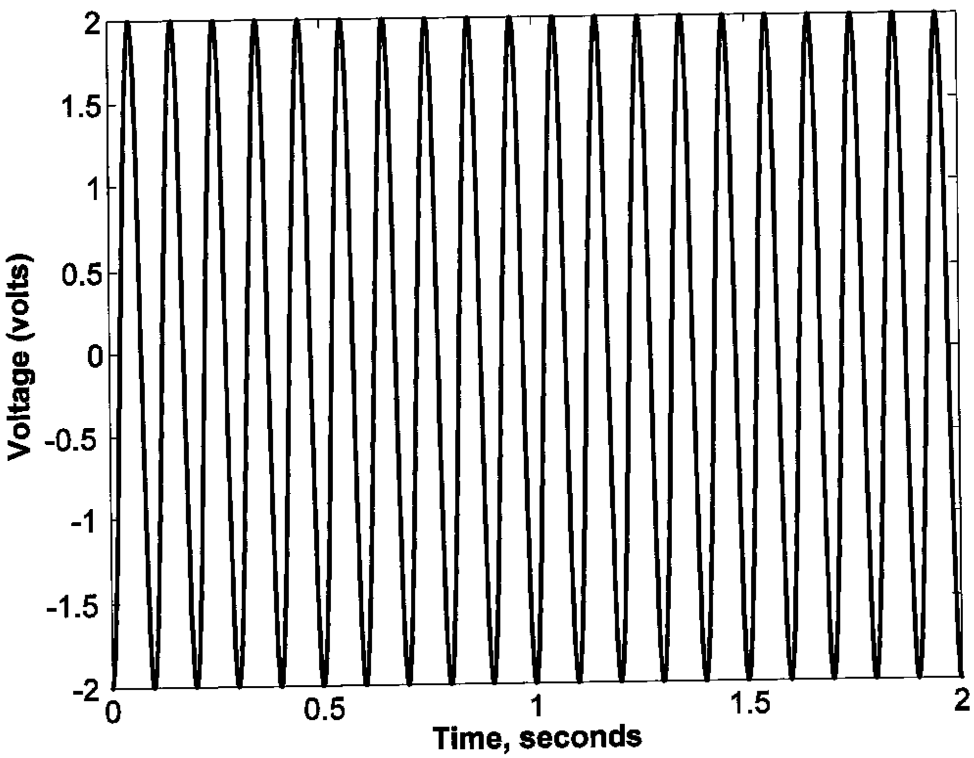

Fig 2a

Frequency Domain

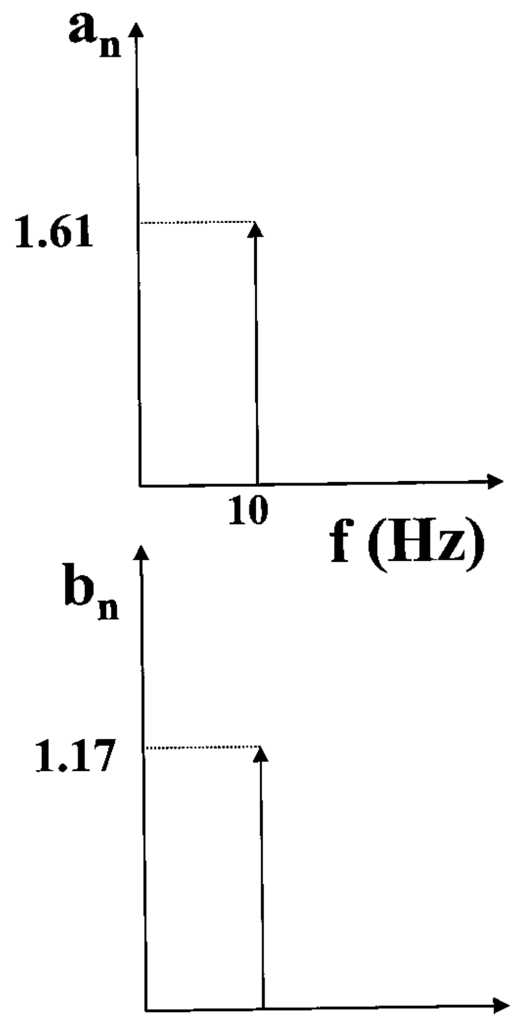

Fig. 2b

$V_2(t)$

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Now, let us add the outputs of two function generators :

$$V(t) = V_1(t) + V_2(t)$$

$$= 5\cos(2\pi \times 5t) + 2\cos[2\pi \times 10 \times (t - 1.11)] \rightarrow (4)$$

Figs. 3a and 3b shows the signal V(t) in time domain and frequency domain, respectively. Note that the period of V(t) is determined by the fundamental frequency, 5 Hz. ie period = 0.2 s. Frequency domain representation of V(t) is obtained by simply adding the amplitudes of sine and cosine components at 5 Hz & 10 Hz.

Time Domain

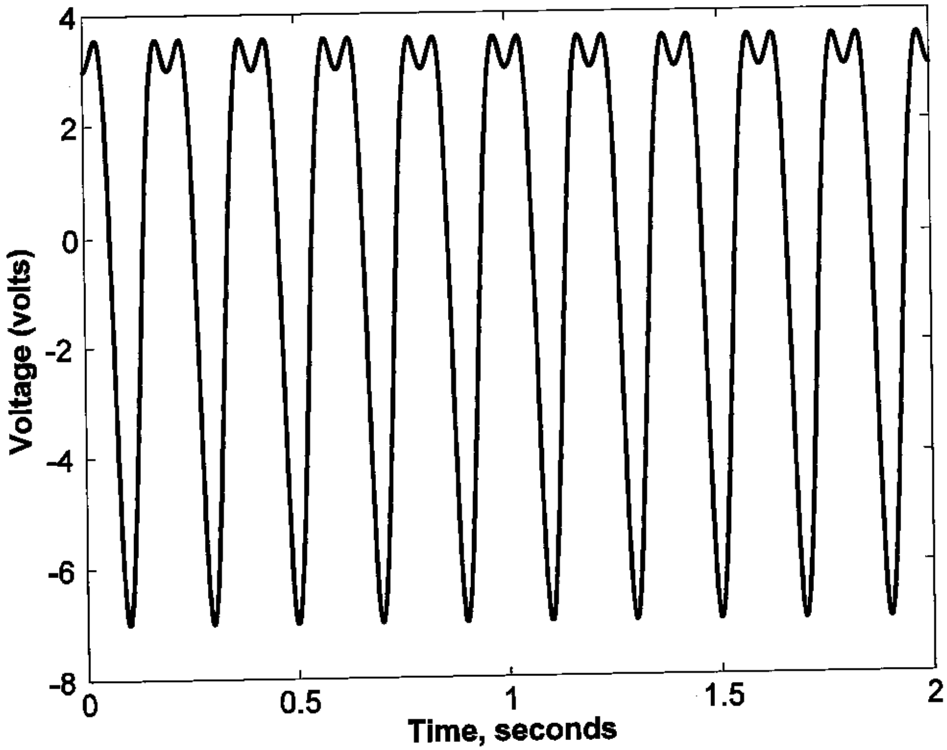
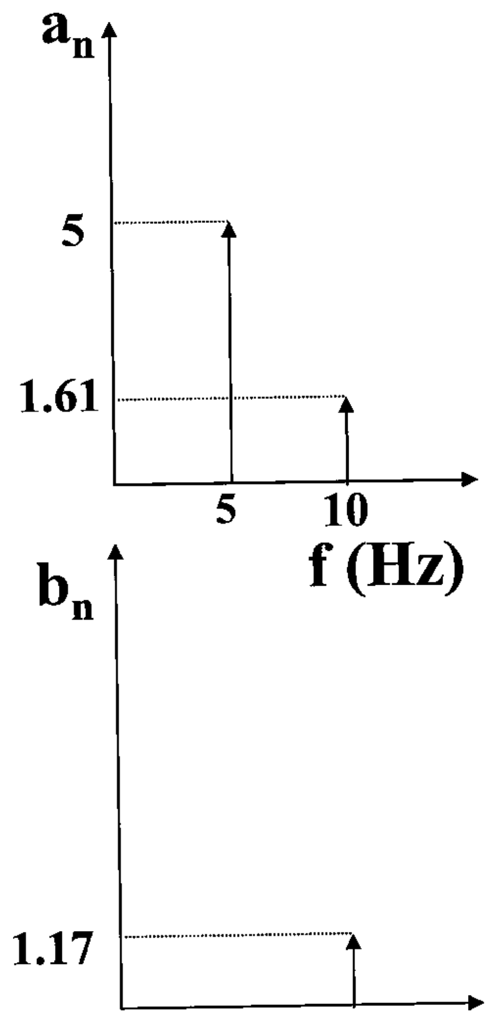


Fig. 3a

Frequency Domain



V(t)

Fig. 3b

To obtain a more complex signal, Let us consider the output of a function generator, $V_3(t)$, whose frequency is a third harmonic of the output of the first function generator, i.e.

$$V_3(t) = 1 \cos\left[2\pi \times 15\,(t-2.0)\right] \longrightarrow (5)$$

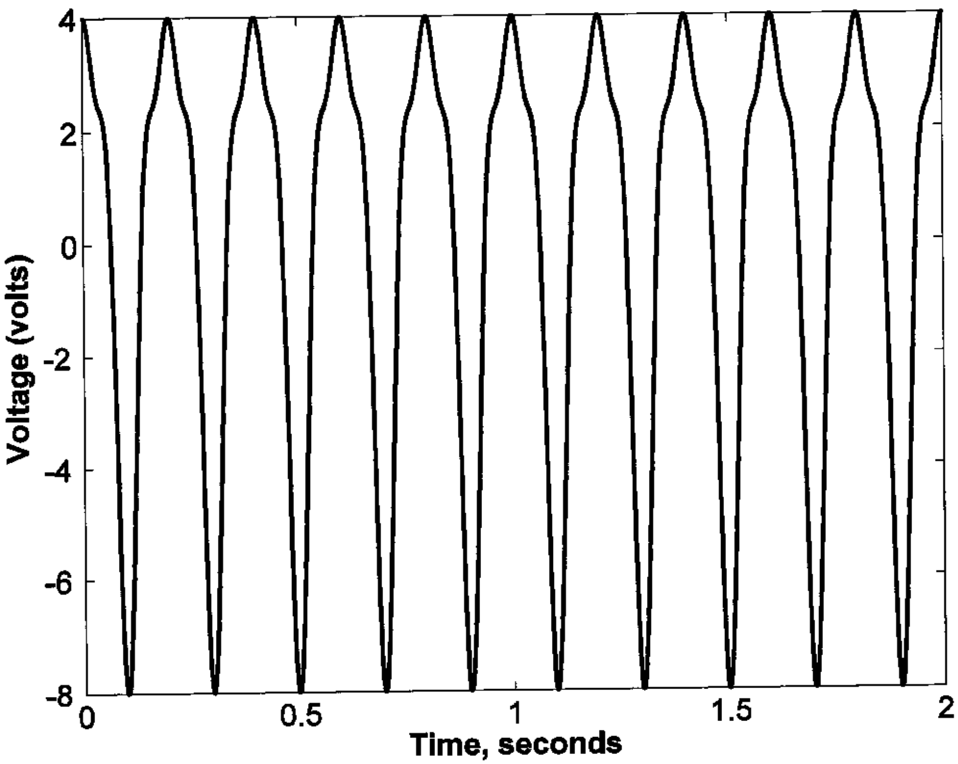
After adding the outputs of all the generators, we have

$$V'(t) = V_1(t) + V_2(t) + V_3(t) \longrightarrow (6)$$

Figs 4a and 4b show the time & frequency domain representation of $V'(t)$, respectively. Notice that the period of the signal $V'(t)$ is still 0.2s which is the inverse of the fundamental frequency, 5 Hz.

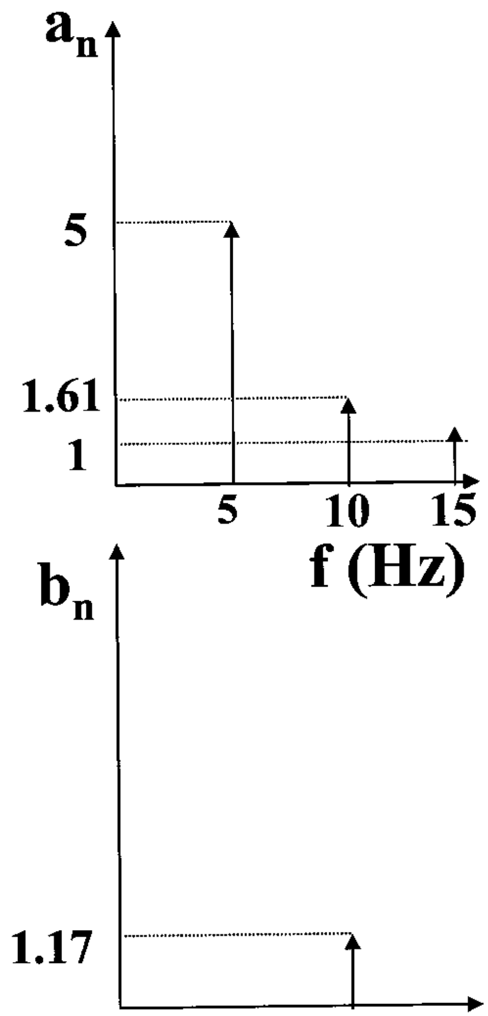
* When a periodic signal with frequency, f_0 and its higher harmonics (nf_0, n = integer) are added to generate a complex periodic signal, the period of the complex signal is still given by $1/f_0$.

Time Domain



Flh. 4a

Frequency Domain



V'(t)

Flh. 4b

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NOW LET US CONSIDER THE INVERSE PROBLEM:

CAN WE REPRESENT AN ARBITRARY PERIODIC SIGNAL AS A SUPERPOSITION OF SINE AND COSINE COMPONENTS?

IN 1878, J.B.J. FOURIER SHOWED THAT IT IS INDEED POSSIBLE. FOURIER WAS WORKING ON HEAT PROPAGATION IN SOLIDS & REALIZED THAT ANY PERIODIC SIGNAL $g_p(t)$, WITH PERIOD T_0 CAN BE EXPRESSED AS A WEIGHTED SUM OF SINES AND COSINES AT HARMONIC FREQUENCIES

$0, 1/T_0, 2/T_0 \ldots$, i.e.,

$$g_p(t) = a_0 + 2\sum_{n=1}^{\infty} a_n \cos\frac{2\pi n t}{T_0} + b_n \sin\frac{2\pi n t}{T_0} \quad (7)$$

DEFINE THE FUNDAMENTAL FREQUENCY f_0 AS $1/T_0$.

THE FUNCTIONS $\cos\left(\frac{2\pi n t}{T_0}\right)$ AND $\sin\left(\frac{2\pi n t}{T_0}\right)$ ARE ~~CALLED~~ KNOWN AS <u>BASIS FUNCTIONS</u>.

PREVIOUSLY, WE CONSTRUCTED A COMPLEX PERIODIC SIGNAL ~~BY ADDING~~ WHEN a_n & b_n ARE KNOWN. NOW, WE ARE LOOKING AT THE INVERSE PROBLEM:

FOR THE GIVEN PERIODIC SIGNAL $g_p(t)$, WHAT ARE THE WEIGHTS (a_n, b_n) OF SINE AND COSINE COMPONENTS?

TO FIND (a_n, b_n), WE MAKE USE OF THE FOLLOWING RELATIONSHIPS:

$$\int_{-T_0/2}^{T_0/2} \cos \frac{2\pi m t}{T_0} \cdot \cos \frac{2\pi n t}{T_0} \, dt = \begin{cases} T_0/2 & m = n \\ 0 & m \neq n \end{cases} \tag{8}$$

$$\int_{-T_0/2}^{T_0/2} \cos \frac{2\pi m t}{T_0} \cdot \sin \frac{2\pi n t}{T_0} \, dt = 0 \quad \text{all } m, n \text{ integer} \tag{9}$$

$$\int_{-T_0/2}^{T_0/2} \sin \frac{2\pi m t}{T_0} \cdot \sin \frac{2\pi n t}{T_0} \, dt = \begin{cases} T_0/2 & m = n \\ 0 & m \neq n \end{cases} \tag{10}$$

TO FIND THE COEFFICIENT a_m, MULTIPLY BOTH SIDES OF (7) BY $\cos \frac{2\pi m t}{T_0}$ AND INTEGRATE:

$$\int_{-T_0/2}^{T_0/2} g_p(t) \cos \frac{2\pi m t}{T_0}\, dt = \int_{-T_0/2}^{T_0/2} a_0 \cos \frac{2\pi m t}{T_0} \cdot dt$$

$$+ 2 \sum_{n=1}^{\infty} \int_{-T_0/2}^{T_0/2} a_n \cos \frac{2\pi n t}{T_0} \cdot \cos \frac{2\pi m t}{T_0}\, dt$$

$$+ 2 \sum_{n=1}^{\infty} \int_{-T_0/2}^{T_0/2} b_n \sin \frac{2\pi n t}{T_0} \cdot \cos \frac{2\pi m t}{T_0}\, dt \rightarrow (11)$$

BY (9), THE LAST TERM IS ZERO. THE FIRST TERM IS ZERO BY (8). ALSO BY (8), THE MIDDLE TERM OF (11) IS ZERO EXCEPT WHEN $n = m$. THEREFORE, WE HAVE

$$\int_{-T_0/2}^{T_0/2} g_p(t) \cdot \cos \frac{2\pi m t}{T_0}\, dt = 2 a_m \cdot T_0/2$$

OR

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_p(t)\cos\frac{2\pi m t}{T_0}\cdot dt = a_m \rightarrow (12)$$

SIMILARLY, MULTIPLYING (7) BY $\sin\frac{2\pi m t}{T_0}$ &

INTEGRATING, WE HAVE

$$b_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_p(t)\cdot\sin\left(\frac{2\pi n t}{T_0}\right)dt \rightarrow (13)$$

INTEGRATING (7), WE OBTAIN

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_p(t)\,dt \rightarrow (14)$$

NOTE THAT a_0 IS SIMPLY THE DC TERM OF

THE SIGNAL.

The Complex Fourier Series

Eqs. (12-14) are a bit awkward in representing the freq. domain. Instead we can represent the freq. domain in a much more compact way by using complex exponentials as basis functions. We show this as follows:

$$\cos\frac{2\pi nt}{T_0} = \frac{1}{2}\left[\exp\left(j\frac{2\pi nt}{T_0}\right) + \exp\left(-j\frac{2\pi nt}{T_0}\right)\right] \longrightarrow (15)$$

$$\sin\frac{2\pi nt}{T_0} = \frac{1}{2j}\left[\exp\left(j\frac{2\pi nt}{T_0}\right) - \exp\left(-j\frac{2\pi nt}{T_0}\right)\right] \longrightarrow (16)$$

Substituting (15) & (16) into (7) we get

$$g_p(t) = a_0 + \sum_{n=1}^{\infty} a_n\left[\exp\left(j\frac{2\pi nt}{T_0}\right) + \exp\left(-j\frac{2\pi nt}{T_0}\right)\right]$$

$$+ \sum_{n=1}^{\infty} \frac{b_n}{j}\left[\exp\left(j\frac{2\pi nt}{T_0}\right) - \exp\left(-j\frac{2\pi nt}{T_0}\right)\right] \longrightarrow (17)$$

Grouping the +ve and −ve exponentials together, we get

$$g_p(t) = a_0 + \sum_{n=1}^{\infty} (a_n - jb_n)\exp\left(j\frac{2\pi nt}{T_0}\right)$$

$$+ \sum_{n=1}^{\infty} (a_n + jb_n)\exp\left(-j\frac{2\pi nt}{T_0}\right) \longrightarrow (18)$$

By putting $n = -m$ in the last term of eq. (18), we obtain

$$\sum_{n=1}^{\infty} (a_n + j b_n) \exp\left(-j \frac{2\pi n t}{T_0}\right) = \sum_{m=-\infty}^{-1} (a_{-m} + j b_{-m}) \exp\left(j \frac{2\pi m t}{T_0}\right)$$

$$= \sum_{m=-\infty}^{-1} (a_m - j b_m) \exp\left(j \frac{2\pi m t}{T_0}\right) \rightarrow (19)$$

Where we define $a_{-m} \triangleq a_m$ and $b_{-m} \triangleq -b_m$.

Combining (19) & (18), we have

$$g_p(t) = a_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (a_n - j b_n) \exp\left(j \frac{2\pi n t}{T_0}\right) \longrightarrow (20)$$

Let us define coefficients c_n as follows:

$$c_n = a_n - j b_n \qquad \cancel{n \neq 0} \; n \neq 0 \longrightarrow (21)$$
$$c_n = a_0 \qquad n = 0$$

Then we have

$$g_p(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(j \frac{2\pi n t}{T_0}\right) \longrightarrow (22)$$

Multiplying eq. (22) by $\exp\left(-j \frac{2\pi m t}{T_0}\right)$ & integrating from $-\frac{T_0}{2}$ to $\frac{T_0}{2}$, it can be shown that.

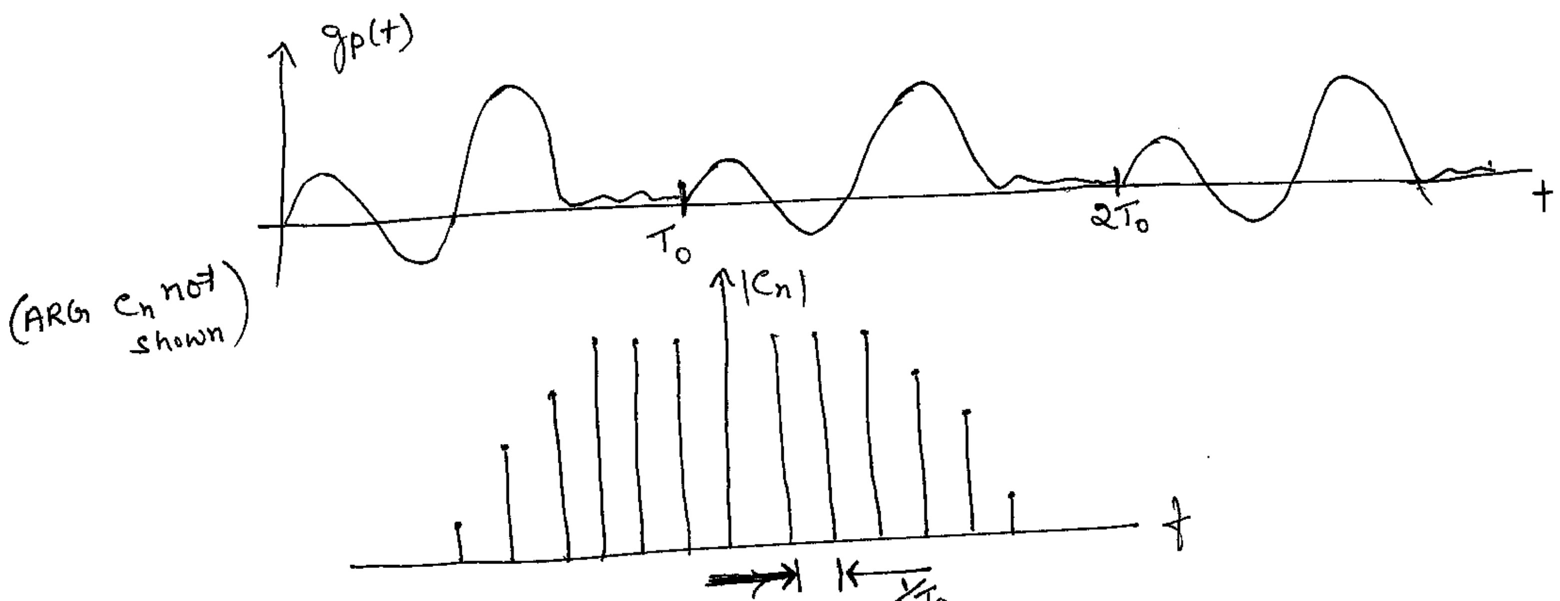
$$C_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} g_p(t) \exp\left(-j\frac{2\pi n t}{T_0}\right) dt \longrightarrow (23)$$

$$n = -\infty, \ldots + \infty .$$

Thus, we see that (22) & (23) define a Fourier series pair. Eq (22) goes from freq. domain to the time domain. Likewise, eq (23) goes from the time domain to the freq. domain.

Eq. (22) is a much more compact and efficient representation of $g_p(t)$ than eqs. (12) - (14). Notice that with the previous representation, we need both a_n's & b_n's to describe $g_p(t)$. However with (22), we only need one set of coefficients. However, note that these C_n's exist for both +ve & −ve n, whereas the a_n, b_n's exist only for +ve n.

Thus, a Fourier series represtation of a function $g_p(t)$ can be depicted as follows:



- Notice that the c_n coefficients are complex. Hence, to specify the frequency domain, we must draw both the magnitude and phase of the c_n coefficients vs. frequency.

- Also, because $b_{-n} = -b_n$, and from (21), we note that $c_n = c_{-n}^*$. Thus, c_n and c_{-n} are a complex conjugate pair.

- The function $g_p(t)$ or the reference c_n are both equally valid means of representing the signal. Sometimes it is more convenient to use $g_p(t)$; in other situations it is easier to deal with the freq. domain representation c_n.

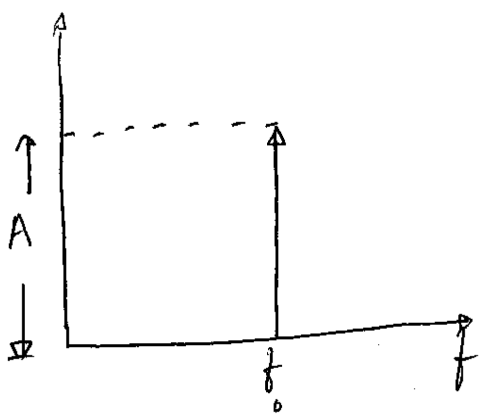
The fact that C_n exists for -ve values of n is somewhat disturbing because the frequency (number of repetitions per second) is a +ve quantity.

What is the meaning of -ve frequency?

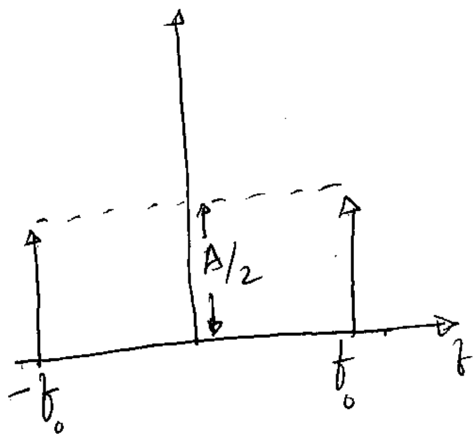
Observe that

$$A \cos(2\pi f_0 t) = \left[\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \right] A$$

In complex fourier series, the signal component at frequency f_0 is represented as components at f_0 and $-f_0$ with half the amplitude as shown below



Real Fourier Series



Complex Fourier Series.

For Real signals, $C_n = C_{-n}^*$, Let us write them as,

$$C_n = A_n e^{i\theta_n} \qquad C_{-n} = B_n \cdot e^{i\phi_n}$$

$$C_n = C_{-n}^* \Longrightarrow A_n = B_n$$

Or

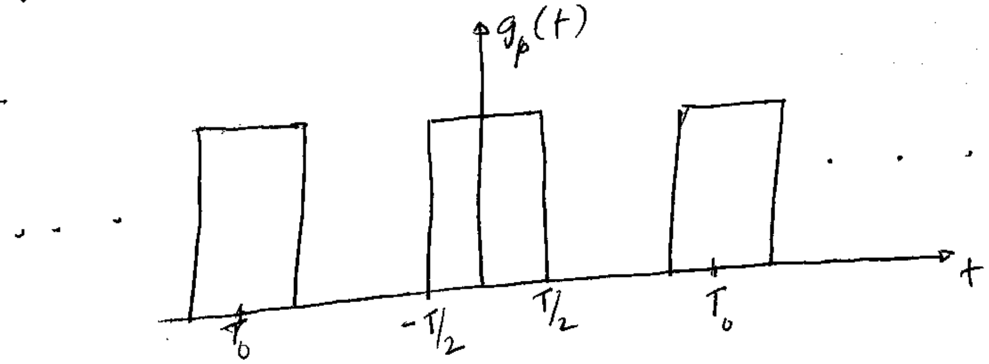
$$|C_n| = |C_{-n}|$$

And $\theta_n = -\phi_n$ or $ARG(C_n) = -ARG(C_{-n})$

Where $ARG(x)$ denotes the phase of x.

Thus, the magnitude spectrum * must be symmetric with f, and the phase spectrum is anti-symmetric with f.

Example 1.



* The spectrum is the function C_n vs. f; ie, the fourier series rep'n of $g_p(t)$. Furthermore, $|C_n|$ vs. f is also called the magnitude frequency response, or just magnitude response. Likewise, ARG C_n is referred to as the phase response.

$$C_n \triangleq \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_p(t)\exp\left(-j\frac{2\pi n t}{T_0}\right)dt$$

$$= \frac{A}{T_0}\int_{-T/2}^{T/2}\exp\left(-j\frac{2\pi n t}{T_0}\right)dt = \frac{-A}{j2\pi n}\, e^{-j2\pi n t/T_0}\Big|_{-T/2}^{T/2}$$

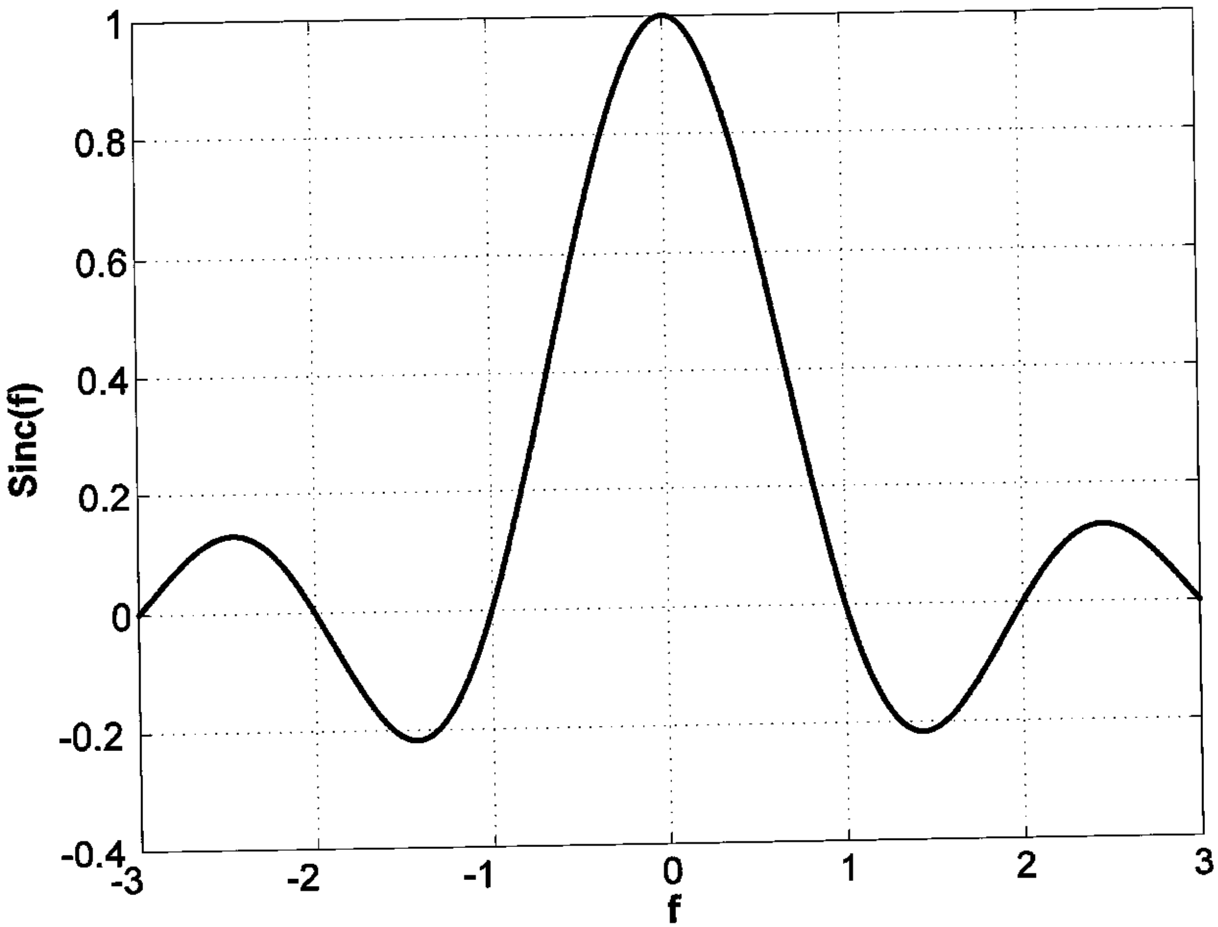
$$= \frac{A}{j2\pi n}\left[e^{j\pi n T/T_0} - e^{-j\pi n T/T_0}\right] = \frac{A}{\pi n}\sin\left(\pi n T/T_0\right)$$

$$= \frac{AT}{T_0}\,\frac{\sin\left(\frac{\pi n T}{T_0}\right)}{\left(\pi n T/T_0\right)}$$

Define the function $\text{sinc}(\lambda) \triangleq \dfrac{\sin(\pi\lambda)}{\pi\lambda}$

Then, $$C_n = \frac{AT}{T_0}\,\text{sinc}\left(\frac{nT}{T_0}\right).$$

Sinc Function



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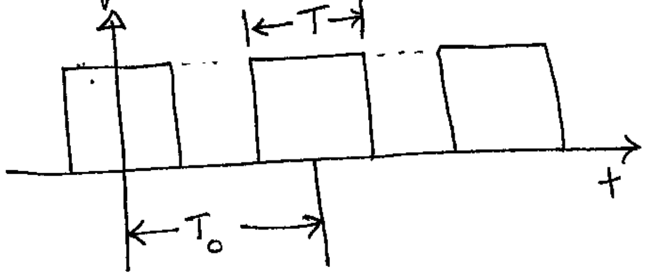
Fourier Series coeffs. for 50% duty cycle square wave



$$g_p(t) = \sum_{n=-\infty}^{\infty} c_n \exp(j 2\pi f_0 n t)$$

$$= \sum_{n=1}^{\infty} 2 c_n \cos(2\pi n f_0 t) + c_0 \qquad (1)$$

where $c_n = \dfrac{AT}{T_0} \operatorname{sinc}\left(\dfrac{nT}{T_0}\right)$

for $\dfrac{T}{T_0} = 1/2$, $A = 1$: $\quad c_n = 1/2 \operatorname{sinc}(n/2)$.

$c_0 = 1$

$2c_1 = 0.6366$ ← coefficients of the cosine term, as in (1) above.

$c_2 = 0$

$2c_3 = -0.2122$ ← The cosine frequency components occur at multiples of $f_0 = 1/T_0$.

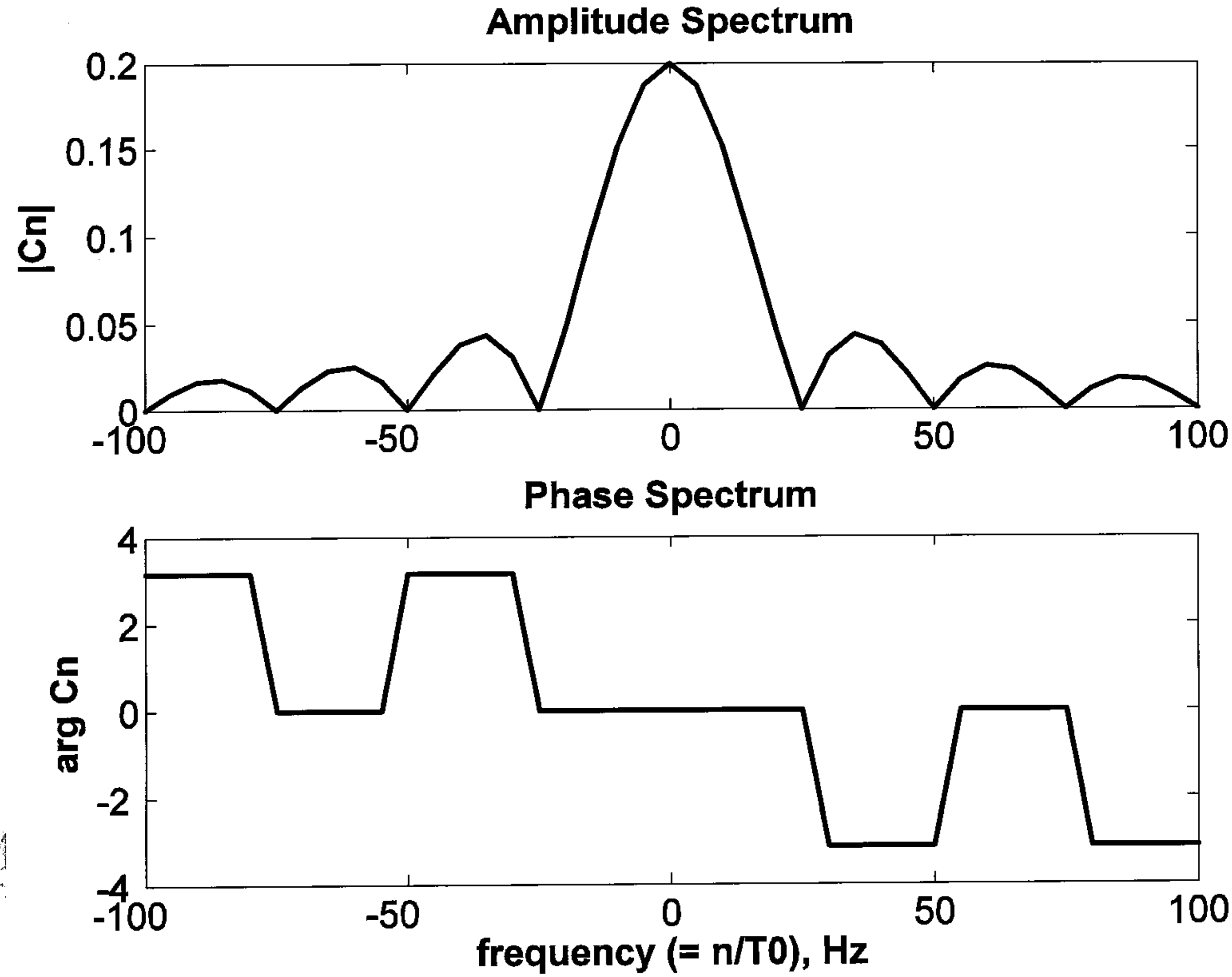
$c_4 = 0$

$2c_5 = 0.1273$

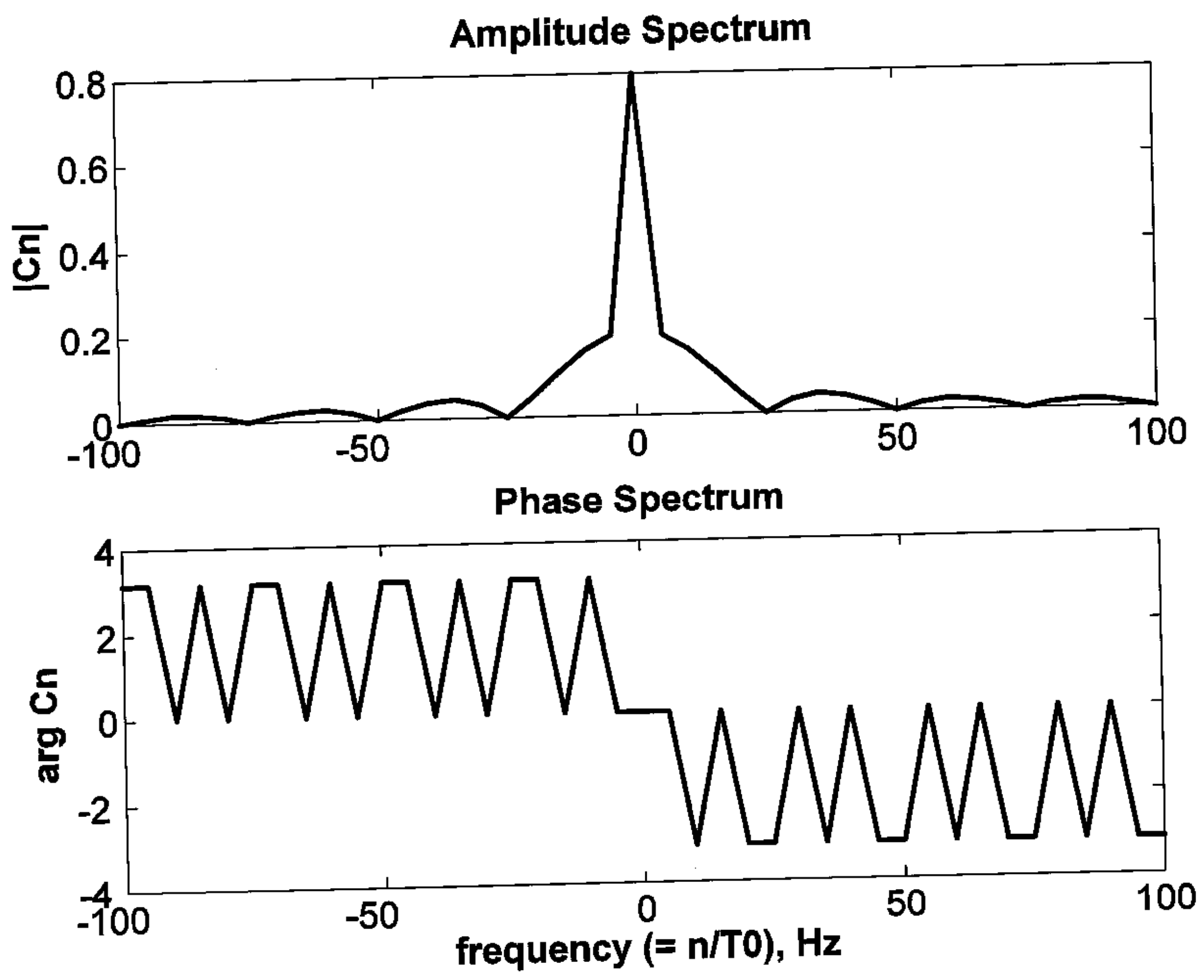
$\vdots$

The following figures show $\tilde{g}_p(t) \triangleq \sum_{n=1}^{N} 2c_n \cos(2\pi n f_0 t) + c_0$ for different values of N.

Spectrum for 20% duty cycle



Spectrum for 80% duty cycle



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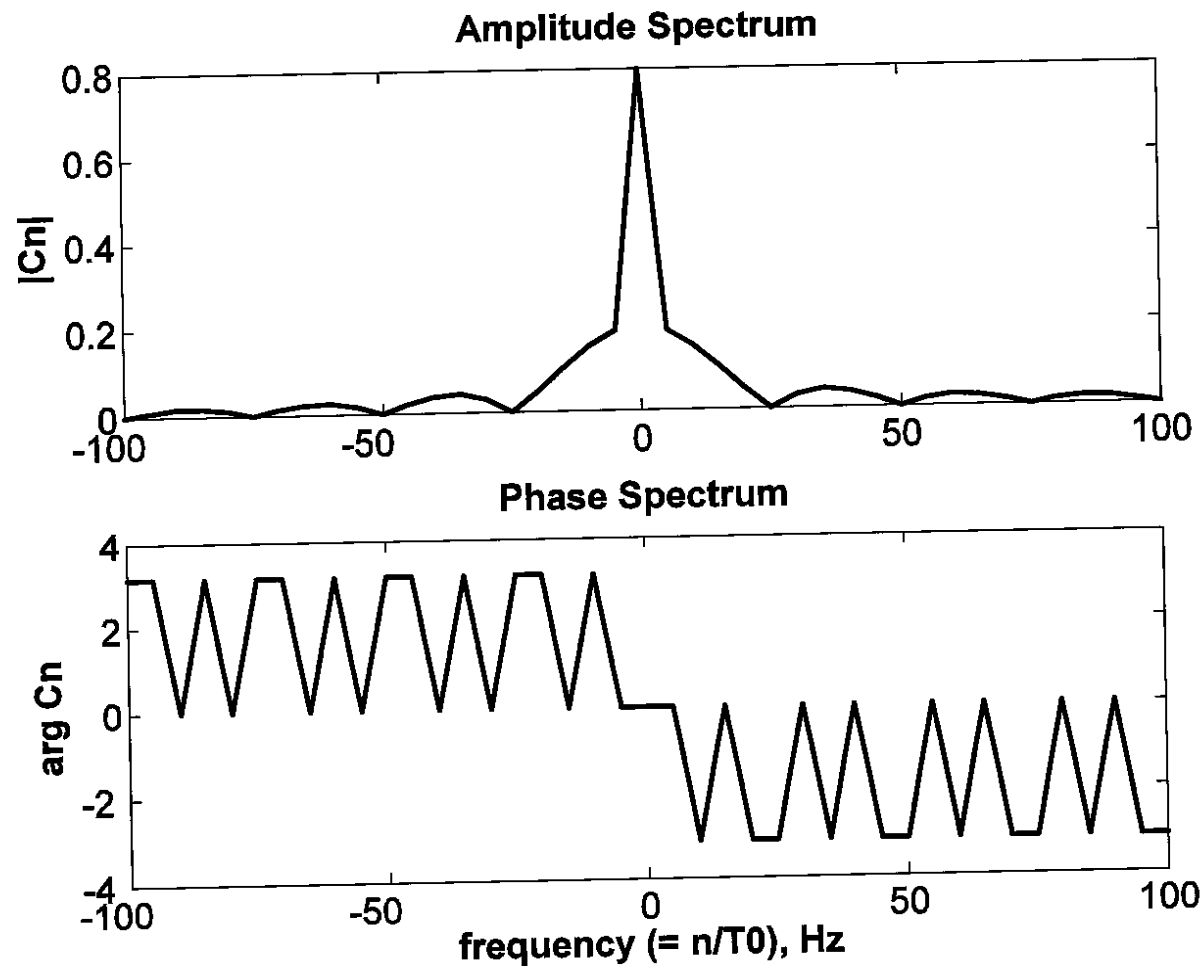
Spectrum for 100% duty cycle



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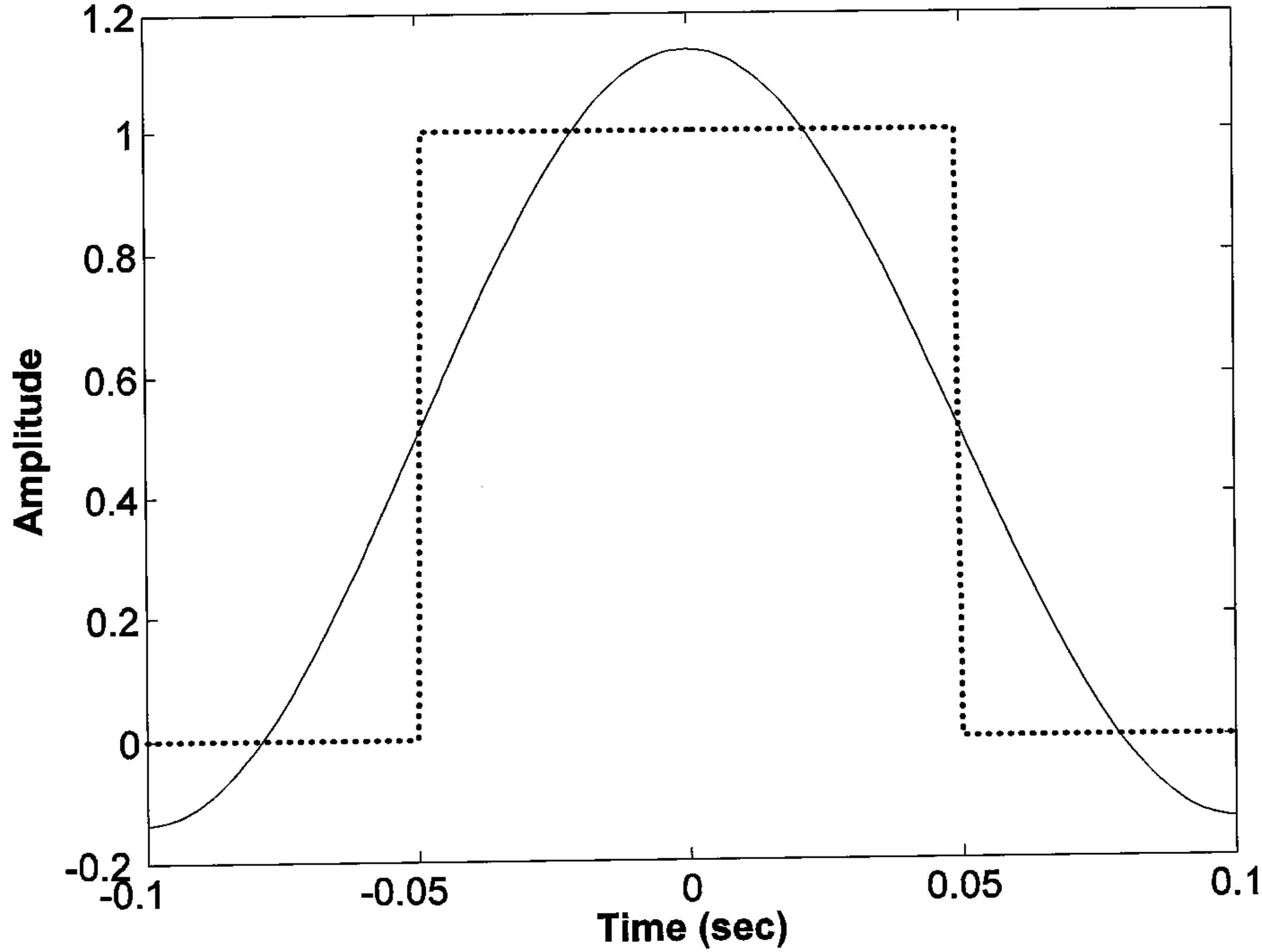
Partial sum for square wave: N=1



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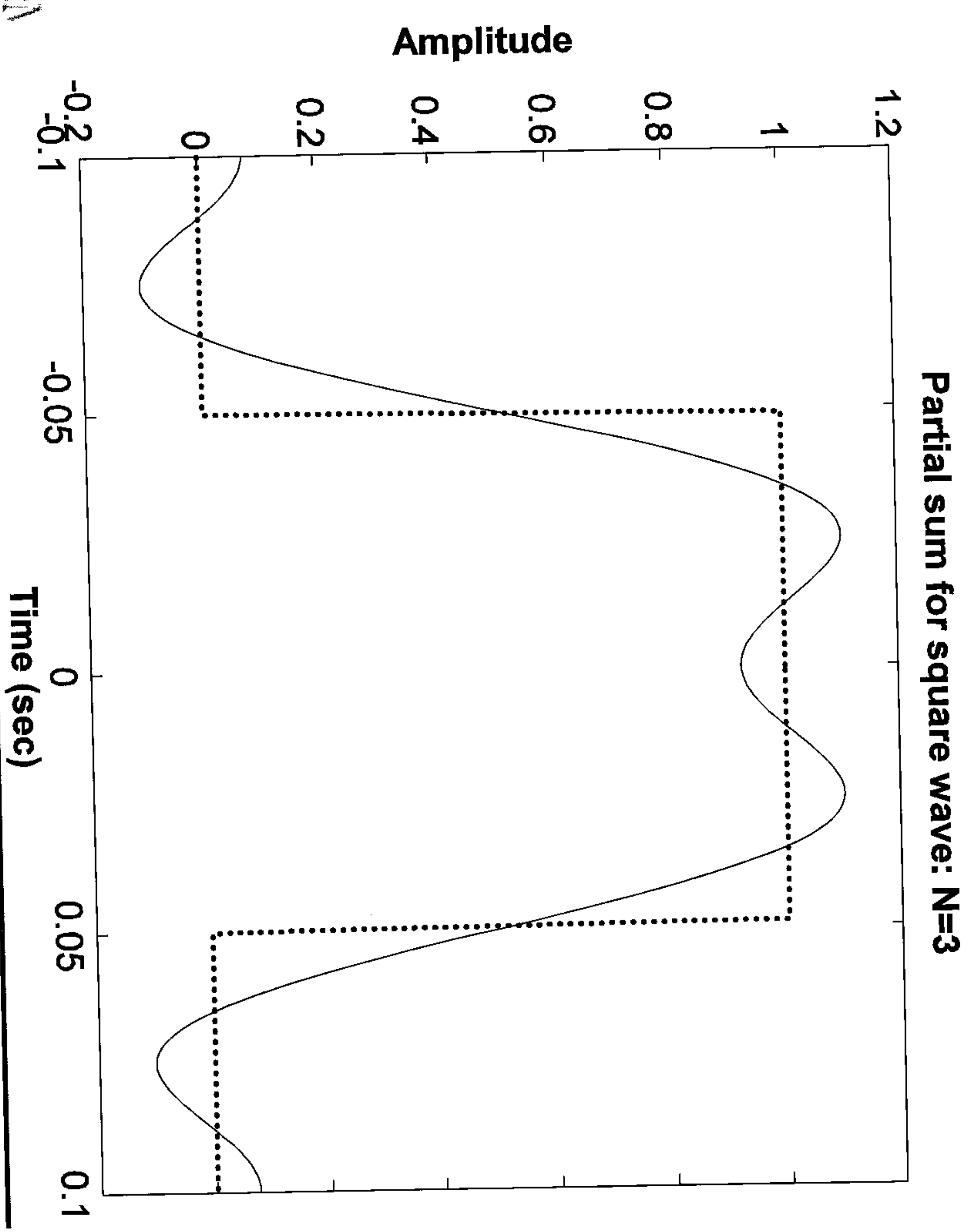
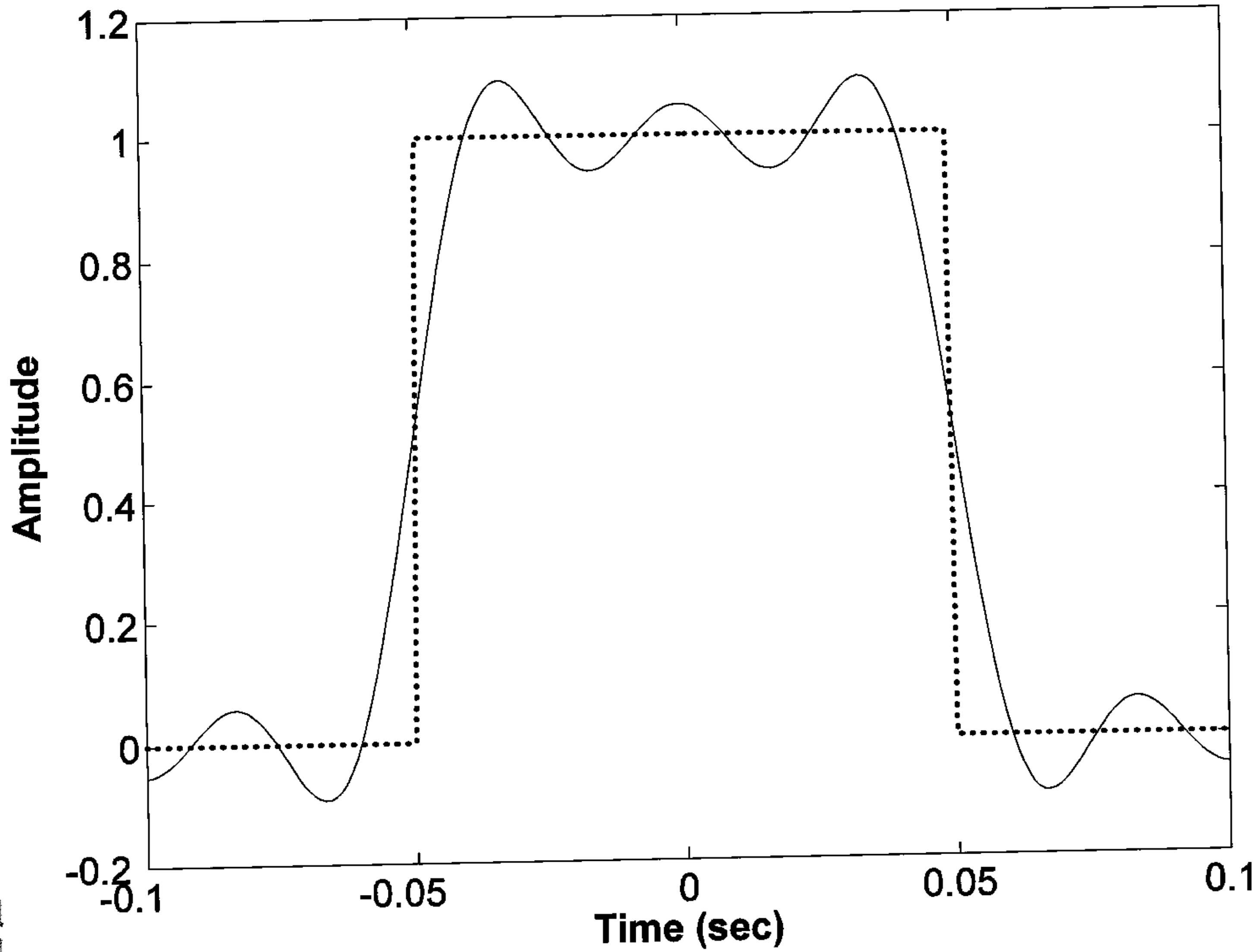

Partial sum for square wave: N=3
Amplitude
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.1
-0.05
0
0.05
0.1
Time (sec)

Partial sum for square wave: N=5



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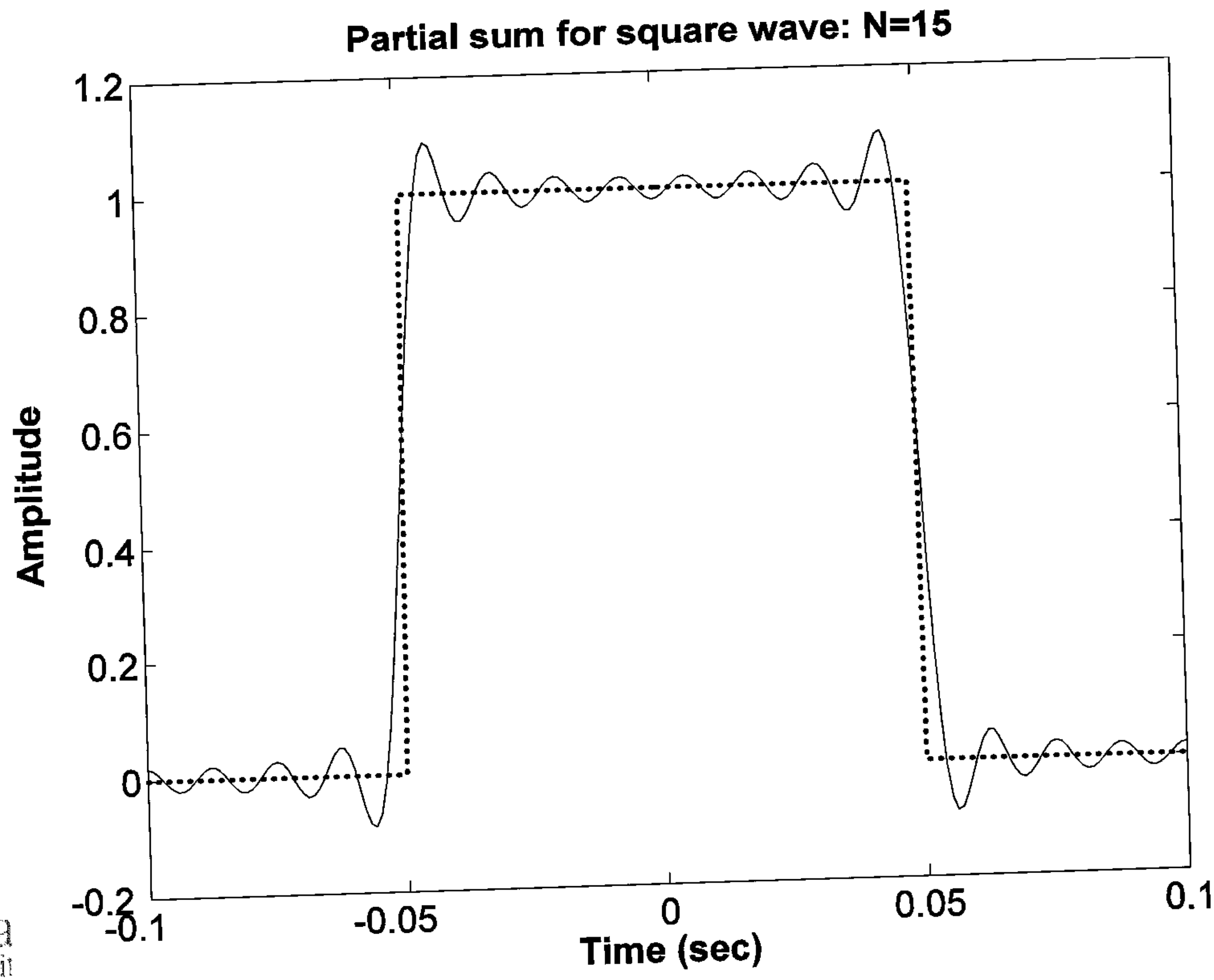

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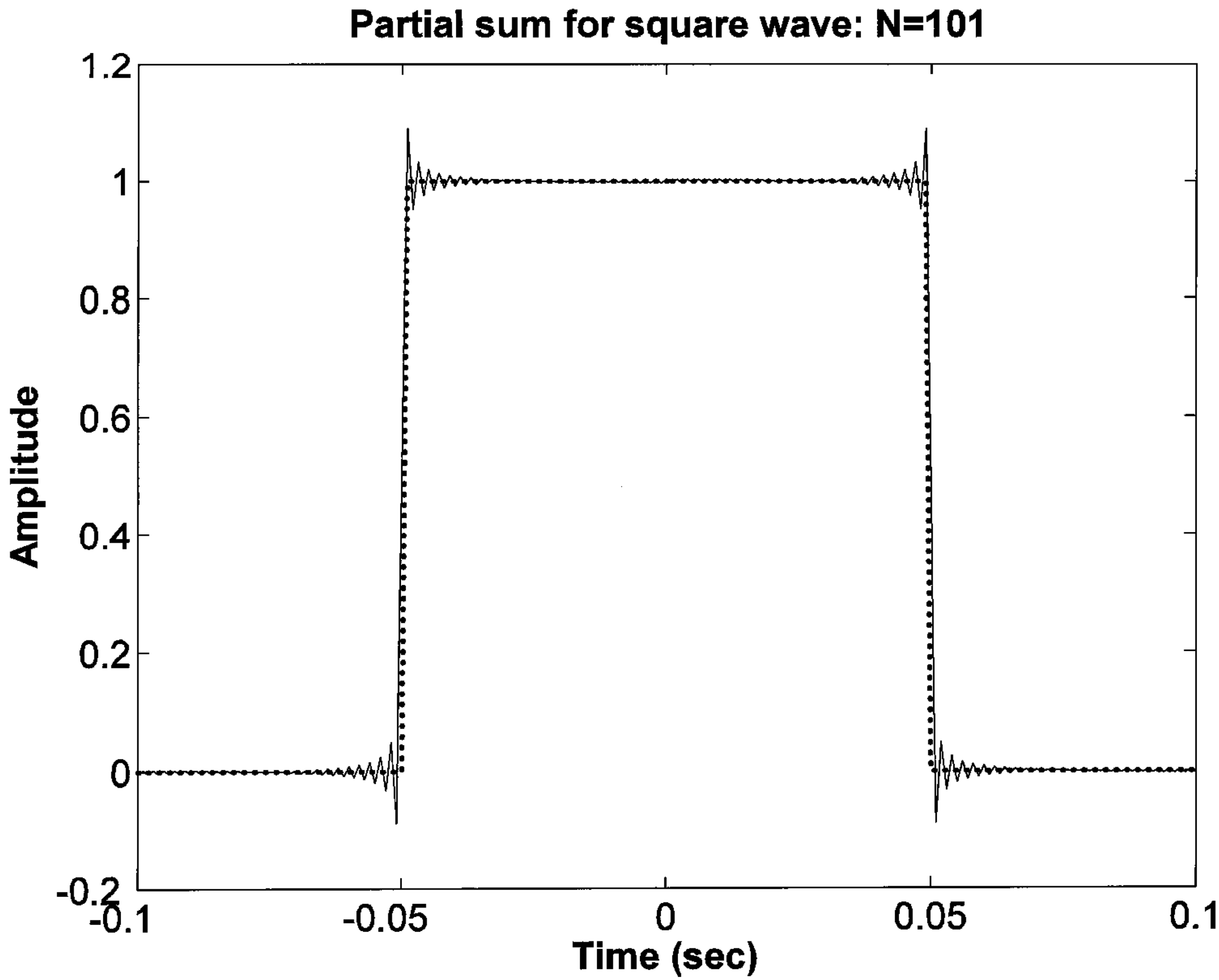
Partial sum for square wave: N=101
Amplitude
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.1
-0.05
0
0.05
0.1
Time (sec)