FOURIER TRANSFORM OF PERIODIC SIGNALS.

EXAMPLE 1:

CONSIDER A DC SIGNAL

$$g_p(t) = 5\text{ V}$$

FOURIER COEFFICIENTS: $C_0 = 5$, other $C_n = 0$

$$C_n = 0 \text{ IF } n \neq 0$$

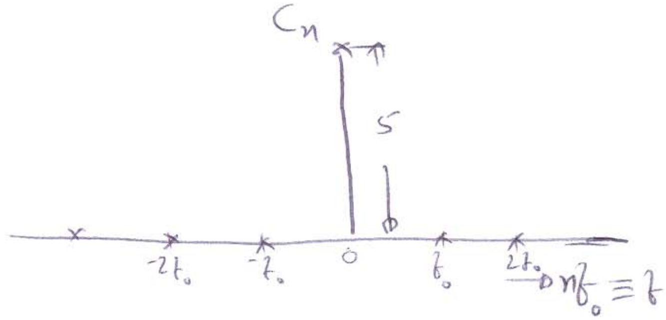


FIG. 1

FOURIER TRANSFORM:

$$\delta(t) \rightleftharpoons 1$$

DUALITY PROPERTY $\to$ $1 \rightleftharpoons \delta(f)$

$$5 \rightleftharpoons 5\delta(f)$$

$$\therefore \mathcal{F}[g_p(t)] = 5\delta(f) = G_p(f)$$

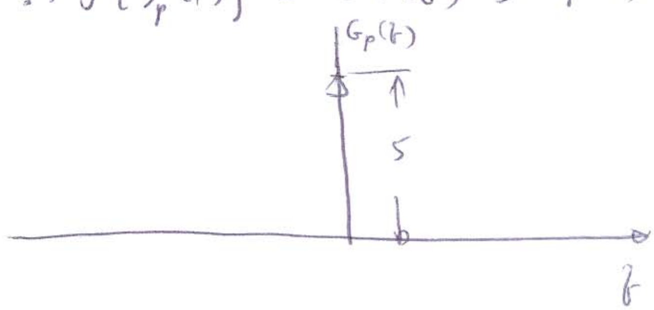


FIG. 2.

COMPARING FIGS. 1 & 2, WE SEE THAT, $G_p(f) = C_0\,\delta(f)$.

EXAMPLE 2 :

$$g_p(t) = 5\cos(2\pi\, 500\, t)$$

FOURIER SERIES : $C_0 = 0$ ($\because$ NO DC COMPONENT)

$f_0 = 500,$ $C_1 = \frac{5}{2}$ ($\because$ WEIGHT OF $\exp(j\, 2\pi 500 t)$)

$C_{-1} = 5/2$ ($\because$ WEIGHT OF $\exp(-j 2\pi 500 t)$)

$C_n = 0$ IF $n \neq 1$.

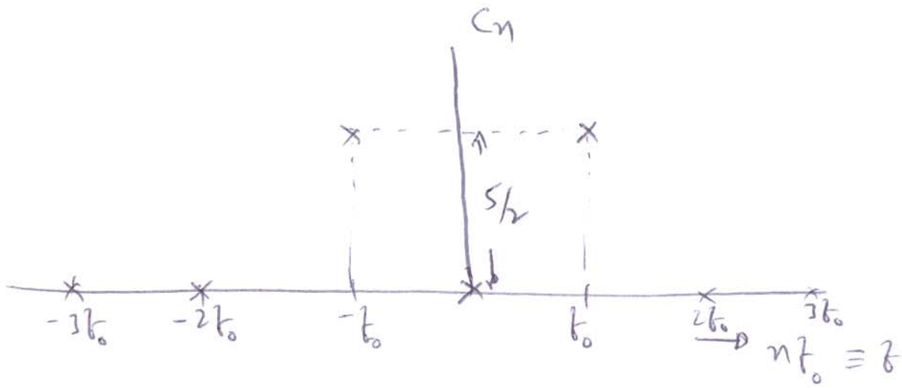


FIG. 3

FOURIER TRANSFORM.

$$1 \rightleftharpoons \delta(f)$$

$$\exp(j500\cdot 2\pi t) \rightleftharpoons \delta(f-500) \rightarrow (1)$$

$$\exp(-j2\pi\cdot 500 t) \rightleftharpoons \delta(f+500) \rightarrow (2)$$

ADDING (1) & (2), $\cos(2\pi\, 500 t) \rightleftharpoons \frac{1}{2}\delta(f-500) + \frac{1}{2}\delta(f+500)$

$$5\cos(2\pi\, 500 t) \rightleftharpoons \frac{5}{2}\delta(f-500) + \frac{5}{2}\delta(f+500)$$

$\left(\because \cos(\theta) = \dfrac{\exp(j\theta) + \exp(-j\theta)}{2}, \ \theta = 2\pi\, 500 t\right)$

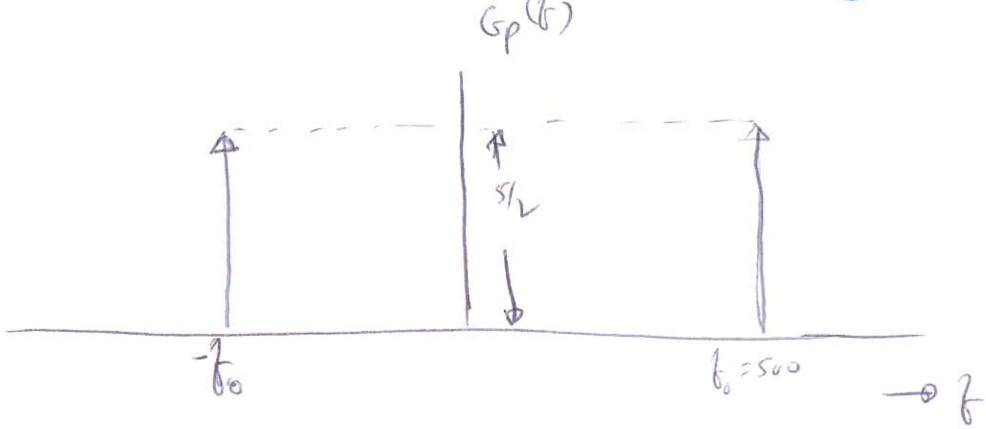


FIG 4.

FROM FIGS. 3 & 4, WE SEE THAT

$$G_p(f) = C_1\,\delta(f - f_0) + C_{-1}\,\delta(f + f_0)$$

EXAMPLE 3:

$$g_p(t) = 5\cos(2\pi\, 500 t) + 2\cos(2\pi\, 1000 t)$$

FOURIER SERIES

$f_0 = 500$ Hz

$2f_0 = 1000$ Hz

$C_1 = 5/2$ (FUNDAMENTAL)

$C_{-1} = 5/2$ (")

$C_2 = 2/2$ (SECOND HARMONIC)

$C_{-2} = 2/2$ (")

$C_n = 0$ IF $n \neq 1$ OR $n \neq 2$

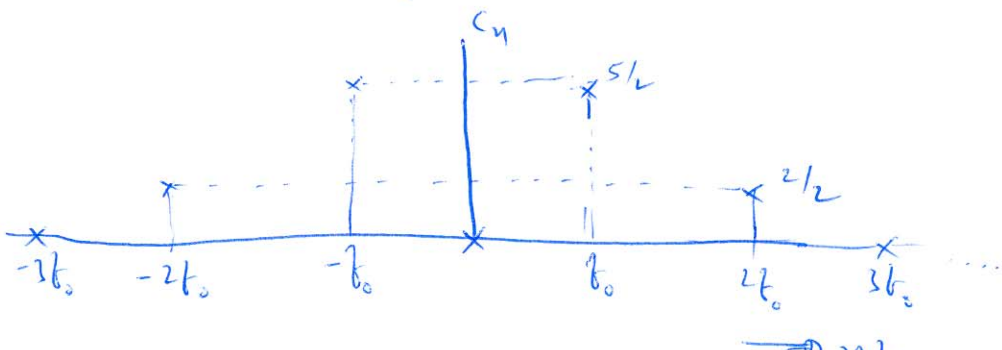


FIG 5

FOURIER SPECTRUM IS A PLOT OF THE (EXP AMPLITUDES OF SINE OR COSINE WAVES (ACTUALLY THAT OF $\exp(j2\pi f t)$) AS A FUNCTION OF THE FREQUENCY.

~~To~~

FOURIER TRANSFORM:

$$5\cos(2\pi\, 500 t) \rightleftharpoons \frac{5}{2}\delta(f-500) + \frac{5}{2}\delta(f+500) \quad \to (3)$$

$$2\cos(2\pi\, 1000 t) \rightleftharpoons \frac{2}{2}\delta(f-1000) + \frac{2}{2}\delta(f+1000) \quad \to (4)$$

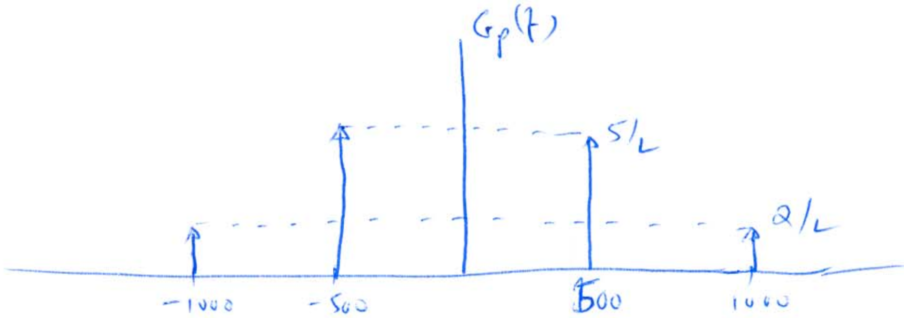


FIG. 6

$$G_p(f) = \frac{5}{2}\delta(f-500) + \frac{5}{2}\delta(f+500) + \frac{2}{2}\delta(f-1000) + \frac{2}{2}\delta(f+1000) \quad \to (5)$$

$$= \cancel{\sum_{n=-\infty}^{\infty} c_n \delta(f-nf_0)}$$

$$= C_1\delta(f-500) + C_{-1}\delta(f+500) + C_2\delta(f-1000) + C_{-2}\delta(f+1000) \quad \to (6)$$

NOTE THE SIMILARITY BETWEEN FOURIER SERIES SPECTRUM (FIG. 5) & FOURIER TRANSFORM SPECTRUM (FIG. 6).

EQ. (6) CAN BE WRITTEN AS

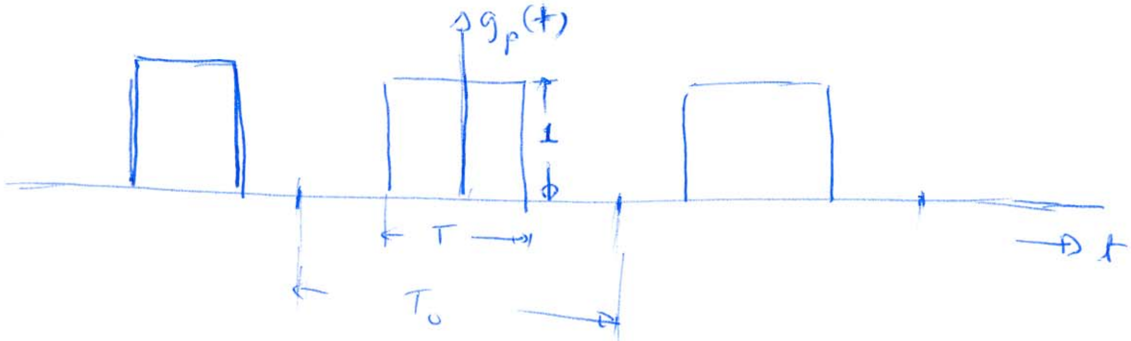
$$G_p(f) = \sum_{n=-2}^{2} C_n \delta(f - nf_0) \quad , \quad f_0 = 500 \text{ Hz}$$

IN GENERAL, IF A PERIODIC SIGNAL ~~HAS~~ $g_p(t)$ HAS A FOURIER COEFFICIENTS C_n, ITS FOURIER TRANSFORM CAN BE WRITTEN AS

$$G_p(f) = \sum_{n=-\infty}^{\infty} C_n \delta(f - nf_0)$$

EXAMPLE 4:

CONSIDER A 50% DUTY CYCLE SQUARE WAVE



$$\text{DUTY CYCLE} = \frac{\text{PULSE WIDTH}}{\text{PERIOD}} \times 100 = \frac{T}{T_0} \times 100$$

$\therefore$ 50% DUTY CYCLE $\Rightarrow$ $T = T_0/2$

$$C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_p(t)\, \exp(-j2\pi n f_0 t)\, dt, \qquad f_0 = 1/T_0$$

$$= \frac{1}{T_0}\int_{-T/2}^{T/2} 1.\ \exp(-j2\pi n f_0 t)\, dt$$

$$= \frac{1}{T_0}\left.\frac{e^{-j2\pi n f_0 t}}{-j2\pi n f_0}\right|_{-T/2}^{T/2} = \frac{1}{2}\,\text{SINC}(n/2)$$

(USE $T = T_0/2$)

$$\text{SINC}(x) = \frac{\sin(\pi x)}{\pi x}.$$

DC COMPONENT, $C_0 = \frac{1}{2}\sin(0) = \frac{1}{2}$

$C_1 = \frac{1}{2}\,\text{SINC}(1/2) = 0.3183$

$C_2 = \frac{1}{2}\,\text{SINC}(2/2) = 0$ ($\because$ SINC$(n) = 0$, n = integer $\neq 0$)

$C_3 = \frac{1}{2}\,\text{SINC}(3/2) = -0.1061$

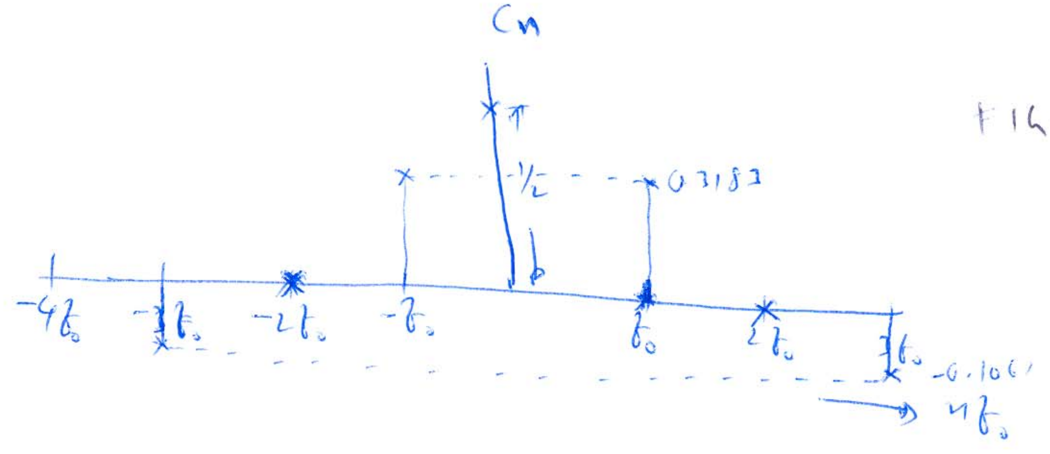


FIG. 7

FOURIER TRASFORM:

$$F[g_p(t)] = G_p(f) = \sum_{n=-\infty}^{\infty} C_n \delta(f - nf_0)$$

$$= \frac{1}{2} \sum_{n=-\infty}^{\infty} \mathrm{sinc}(n/2)\, \delta(f - nf_0)$$

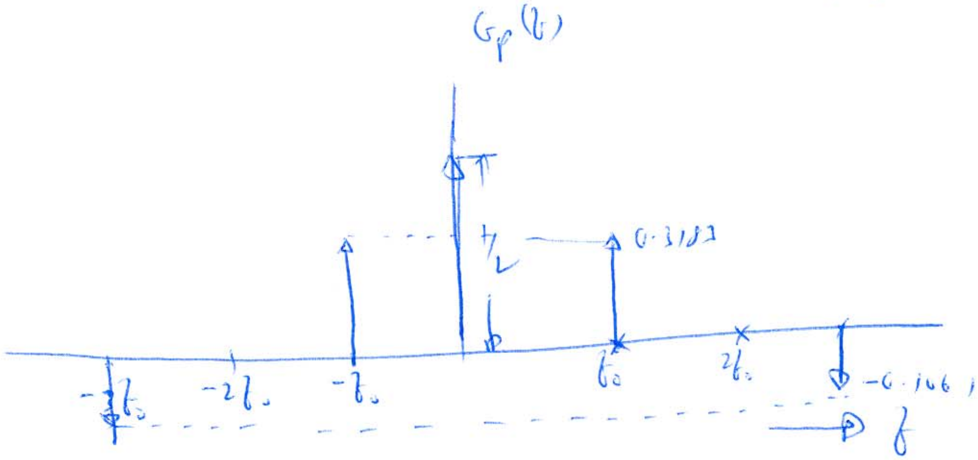
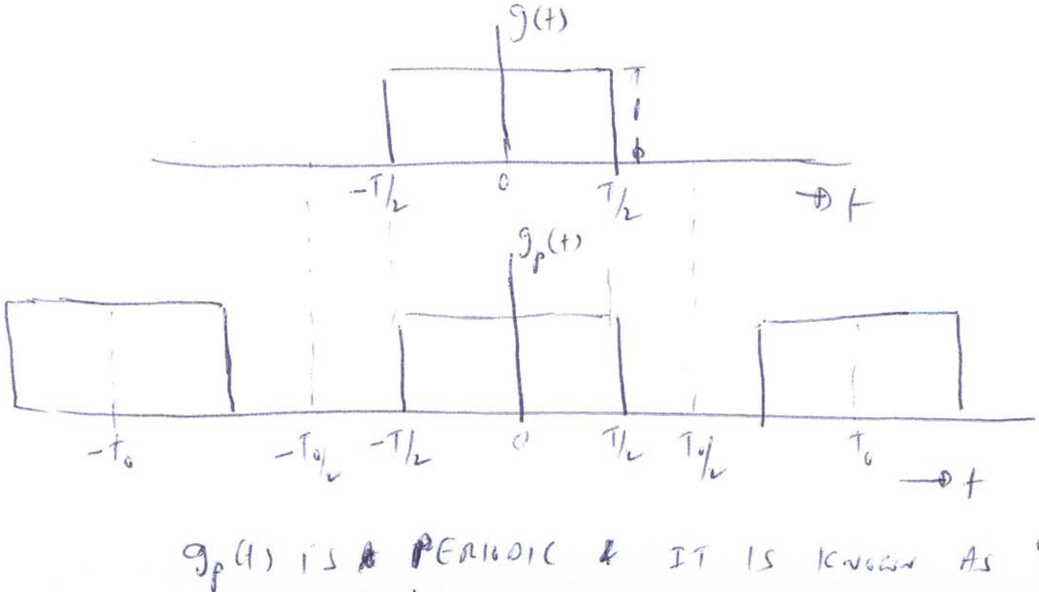


FIG. 8

EXAMPLE 5: IN EXAMPLE 4, CONSIDER THE RECTANGULAR PULSE WITHIN A PERIOD T_0.

$$g(t) = \mathrm{rect}(t/T)$$



$$g_p(t) = \sum_{n=-\infty}^{\infty} g(t - nT_0)$$

$$g_p(t) = \sum_{n=-\infty}^{\infty} \mathrm{rect}\left(\frac{t - nT_0}{T}\right)$$

$g_p(t)$ IS PERIODIC & IT IS KNOWN AS 'PERIODIC EXTENSION OF $g(t)$'.

SINCE $\text{rect}(t) \rightleftharpoons \text{sinc}(f)$

SCALING PROPERTY $\leftarrow$ $\text{rect}(t/T) \rightleftharpoons T\,\text{sinc}(fT)$

$T = T_0/2$

$\therefore G(f) = T\,\text{sinc}(fT) = \frac{T_0}{2}\,\text{sinc}(f\,T_0/2)$

CONSIDER THE FOURIER SPECTRUM OF $g(t)$ at n^{th} harmonic of ~~THE~~ ITS PERIODIC EXTENSION $g_p(t)$.

$$G(f)\Big|_{nf_0} = \frac{T_0}{2}\,\text{sinc}(nf_0 T_0/2) = \frac{T_0}{2}\,\text{sinc}(n/2) \quad (\because f_0 T_0 = 1)$$

SINCE THE FOURIER COEFFICIENTS OF $G_p(f)$ ARE

$$C_n = \frac{1}{2}\,\text{sinc}(n/2)$$

IT FOLLOWS THAT

$$G(f)\Big|_{nf_0} = G(nf_0) = T_0\,C_n \longrightarrow (*)$$

or

$$\boxed{C_n = f_0\,G(nf_0)}$$

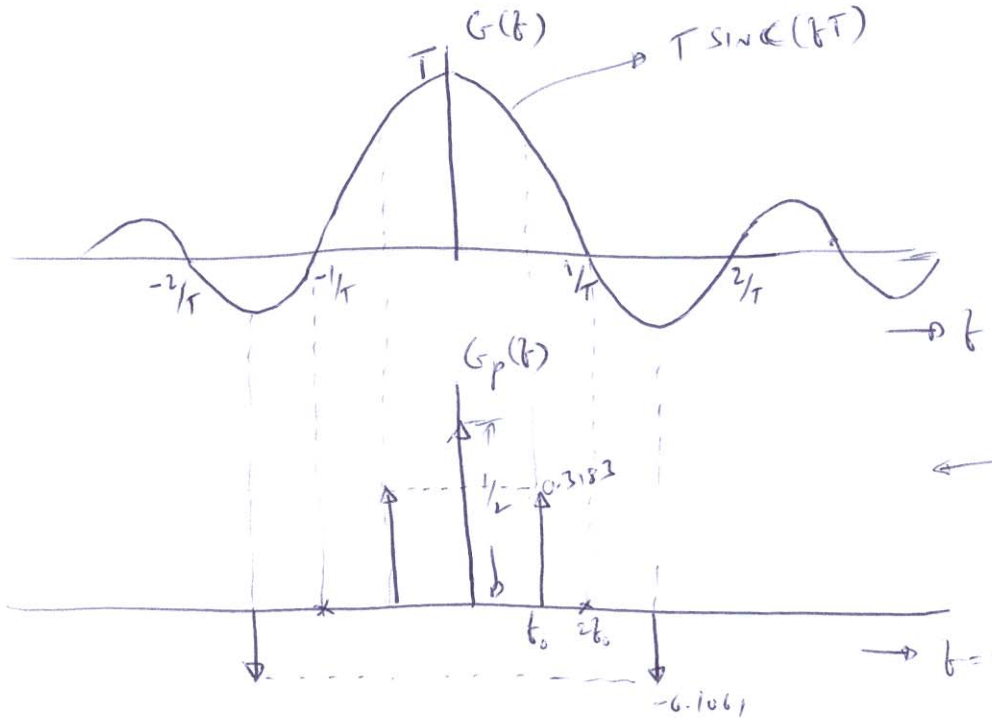
IN EXAMPLE 4, WE HAVE SEEN THAT

$$G_p(f) = \sum_{n=-\infty}^{\infty} C_n\,\delta(f - nf_0) \longrightarrow (**)$$

USING EQ. (*) IN (**), WE FIND

$$\boxed{G_p(f) = f_0 \sum_{n=-\infty}^{\infty} G(nf_0)\,\delta(f-nf_0)} \longrightarrow (***)$$

ALTHOUGH WE PROVED THE ABOVE PROPERTY FOR A PERIODIC WAVE CONSISTING OF RECTANGULAR PULSES, IT HOLDS TRUE FOR ARBITRARY TIME LIMITED SIGNAL $g(t)$ WITHIN A PERIOD T_0.



$T = T_0/2$

FIG. 9

SAME AS FIG. 8

FIG. 10

MEANING OF EQ. (***)

SUPPOSE WE KNOW THE FOURIER TRANSFORM OF A PULSE $g(t)$ $(= G(f))$. ITS PERIODIC EXTENSION $g_p(t)$ WITH PERIOD T_0 CAN BE OBTAINED BY TAKING THE SAMPLES OF $G(f)$ AT $f = nf_0$ & MULTIPLYING THEM BY DELTA FUNCTIONS LOCATED AT $f = nf_0$, AS SHOWN IN FIG. 10.