EE3TR4 Midterm

Feb 28 2005

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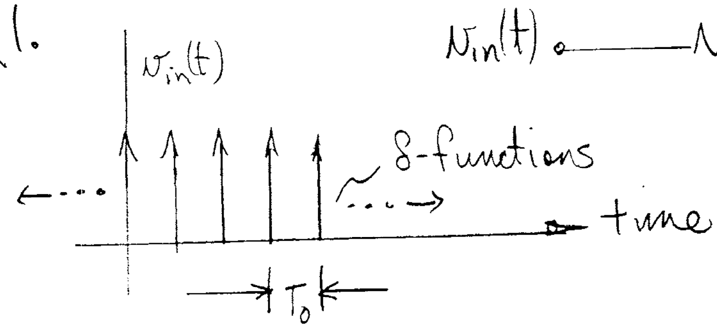
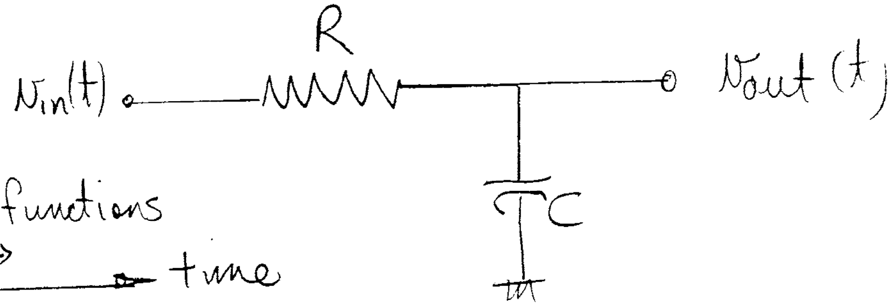
- Answer all questions.
- All major questions are of equal weight.
- This test consists of 3 questions on two sides of a single sheet of paper. Bring any discrepancy to the attention of an invigilator.

Q1.





The input $v_{in}(t)$ is an impulse train described by $v_{in}(t) = \sum_{i=-\infty}^{\infty} \delta(t - iT_0)$ and $RC = T_0/5$.

a) Sketch $v_{out}(t)$ vs. t. Show all relevant values.

b) Sketch $V_{out}(f)$ vs. f. Show all relevant values. Calculate the magnitude and phase of $V_{out}(f)$ at $f = \frac{2}{T_0}$

Hint:

$$\sum_{i=-\infty}^{\infty} \delta(t - iT_0) \rightleftharpoons \frac{1}{T_0} \sum_{i=-\infty}^{\infty} \delta(f - i/T_0) \; ; \quad e^{-at}u(t) \rightleftharpoons \frac{1}{a + j2\pi f}, \; a >$$

Q2.

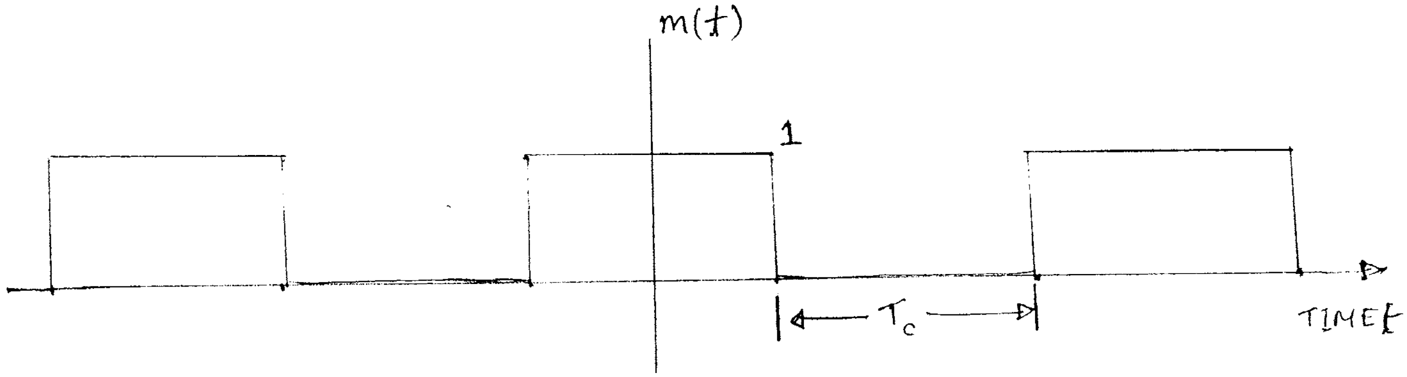
Consider $x(t) = \cos(2\pi f_c t + \theta) + w(t)$

where θ is a random variable with probability density function $p(\theta)$ given by

$$p(\theta) = \begin{cases} 1/2\pi, & 0 \leq \theta \leq 2\pi. \\ 0, & \text{otherwise}, \end{cases}$$

and $w(t)$ is a zero mean white stationary random process, independent of θ, with power spectral density $S_w(f) = 1$. Calculate the autocorrelation function $R_X(\tau)$ and power spectral density $S_X(f)$.

Q3 : CONSIDER THE SQUARE WAVE MESSAGE SIGNAL $m(t)$



a) DRAW THE CORRESPONDING AM MODULATED WAVE FOR 1) 50% MODULATION 2) 100% MODULATION.

NOTE : % MODULATION IS DEFINED AS $\max|k_a m(t)| \times 100$.

b) SKETCH THE SPECTRUM FOR 100% MODULATION SHOWING ALL RELEVANT VALUES. ASSUME $A_c = 1$.

EE 3TR4 Midterm Solutions

1. First, we look at the time domain.

The frequency response $H(f)$ of the filter is:

$$H(f) = \frac{1}{1+j2\pi fRC} \qquad (1)$$

From the hint,

$$\frac{1}{a+j2\pi f} = \frac{1/a}{1+j2\pi f/a} \rightleftharpoons e^{-at}\,u(t) \qquad (2)$$

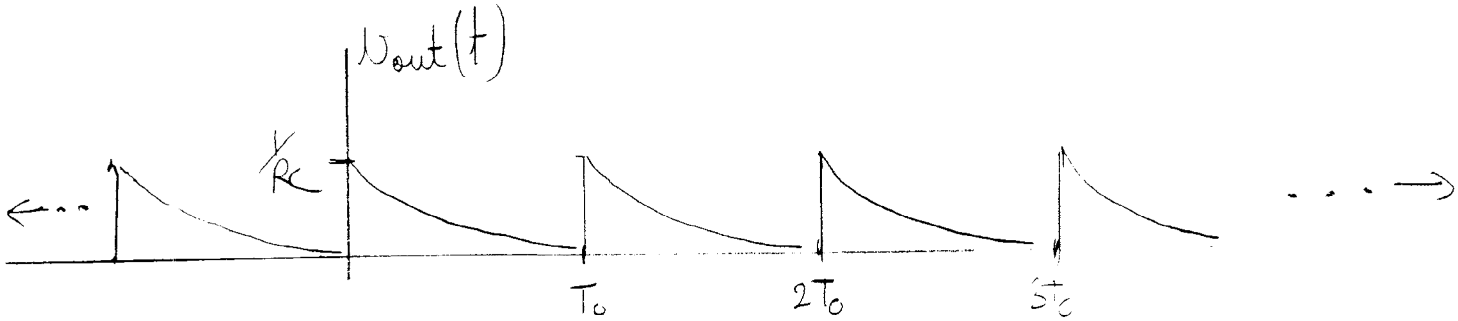
Comparing (1) and (2), it is clear that $a = 1/RC$

Therefore $\frac{1}{1+j2\pi fRC} \rightleftharpoons \frac{1}{RC}\, e^{-t/RC}\, u(t)$

and the impulse response $h(t) = \frac{1}{RC}\, e^{-t/RC}\, u(t)$.

Since $5RC = T_0$, the impulse response has almost decayed to zero within a time period of T_0 secs (i.e $T_0 = 5$ time constants).

The output $v_0(t)$ looks like:



(b) Frequency domain:

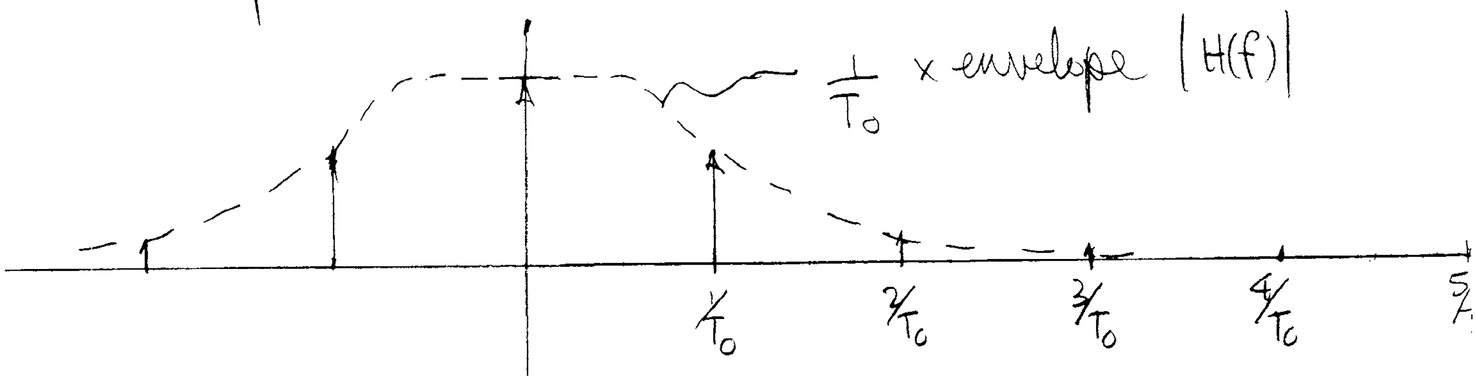
$$V_{out}(f) = H(f) \cdot V_{in}(f)$$

$$= \frac{1}{1+j2\pi f RC} \cdot \left[\frac{1}{T_0} \sum_{i=-\infty}^{\infty} \delta(f - i/T_0) \right]$$

The cutoff of the filter is at $f = \dfrac{1}{2\pi RC}$

$$= \frac{1}{2\pi} \times \frac{5}{T_0} \simeq \frac{0.7958}{T_0}.$$

$\therefore$ The spectrum $V_{out}(f)$ looks like



1b (cont'd)

$$V_{out}(f) = \frac{1}{1+j2\pi fRC} \cdot \frac{1}{T_0} \sum_i \delta(f - i/T_0)$$

at $f = 2/T_0$ we have (recall $RC = T_0/5$)

$$V_{out}(f) = \frac{1}{1+j2\pi\left(\frac{2}{T_0}\right)\left(\frac{T_0}{5}\right)} \cdot \frac{1}{T_0}\,\delta(f - 2/T_0)$$

$$= \frac{1}{1+j\,4\pi/5} \cdot \frac{1}{T_0}\,\delta(f - 2/T_0)$$

$$\cong \frac{1}{1+j\,2.513} \cdot \frac{1}{T_0}\,\delta(f - 2/T_0)$$

$$= \frac{(0.1367 - j0.3435)}{T_0}\,\delta(f - 2/T_0)$$

$$= \frac{0.3697}{T_0}\, e^{-68.30^\circ} \cdot \delta(f - 2/T_0)$$

$\therefore$ $\left|V_{out}(f)\right|_{f=2/T_0} = \dfrac{0.3697}{T_0}$

$\measuredangle V_{out}\,\big|_{f=2/T_0} = -68.30^\circ.$

Q2 Solutions

$$x(t) = \cos(2\pi f_c t + \theta) + w(t)$$

If $w(t)$ has zero mean, $x(t)$ also has zero mean.

$$\therefore R_X(\tau) = E\left(x(t)\, x(t+\tau)\right)$$

$$= E\left(\left[\cos(2\pi f_c t + \theta) + w(t)\right]\left[\cos(2\pi f_c (t+\tau) + \theta) + w(t+\tau)\right]\right)$$

$$= E \cos(2\pi f_c t + \theta) \cos(2\pi f_c (t+\tau) + \theta)$$

$$+ E\left[\cos(2\pi f_c t + \theta) \cdot w(t)\right]$$

$$+ E\left[\cos(2\pi f_c (t+\tau) + \theta) \cdot w(t+\tau)\right]$$

$$+ E\left[w(t)\, w(t+\tau)\right]$$

Since $w(t)$ and θ are independent, the expectations in the middle two terms are zero.

$$\therefore R_X(\tau) = E\left[\cos(2\pi f_c t + \theta) \cos(2\pi f_c (t+\tau) + \theta)\right]$$

$$+ E\left(w(t)\, w(t+\tau)\right).$$

Q2 cont'd,

To evaluate the first term, we use

$$\cos A \cos B = \tfrac{1}{2}\cos(A+B) + \tfrac{1}{2}\cos(A-B)$$

$\therefore$ first term is:

$$E\left(\tfrac{1}{2}\cos\left(2\pi f_c(2t+\tau)+2\Theta\right)\right) + E\left(\tfrac{1}{2}\cos 2\pi f_c \tau\right)$$

$$= \tfrac{1}{2}\cos 2\pi f_c \tau.$$

The second term is the autocorrelation of the noise. Since $w(t)$ is white, with PSD $=1$, the 2nd term is $\delta(\tau)$.

$$\therefore R_X(\tau) = \tfrac{1}{2}\cos 2\pi f_c\tau + \delta(\tau).$$

Q3: THE AM WAVE IS GIVEN BY

$$s(t) = [1 + k_a m(t)] \cos(2\pi f_c t) \quad \text{(3.1)}$$

(a) 1) 50% MODULATION:

$$\text{MAX} [m(t)] = 1$$

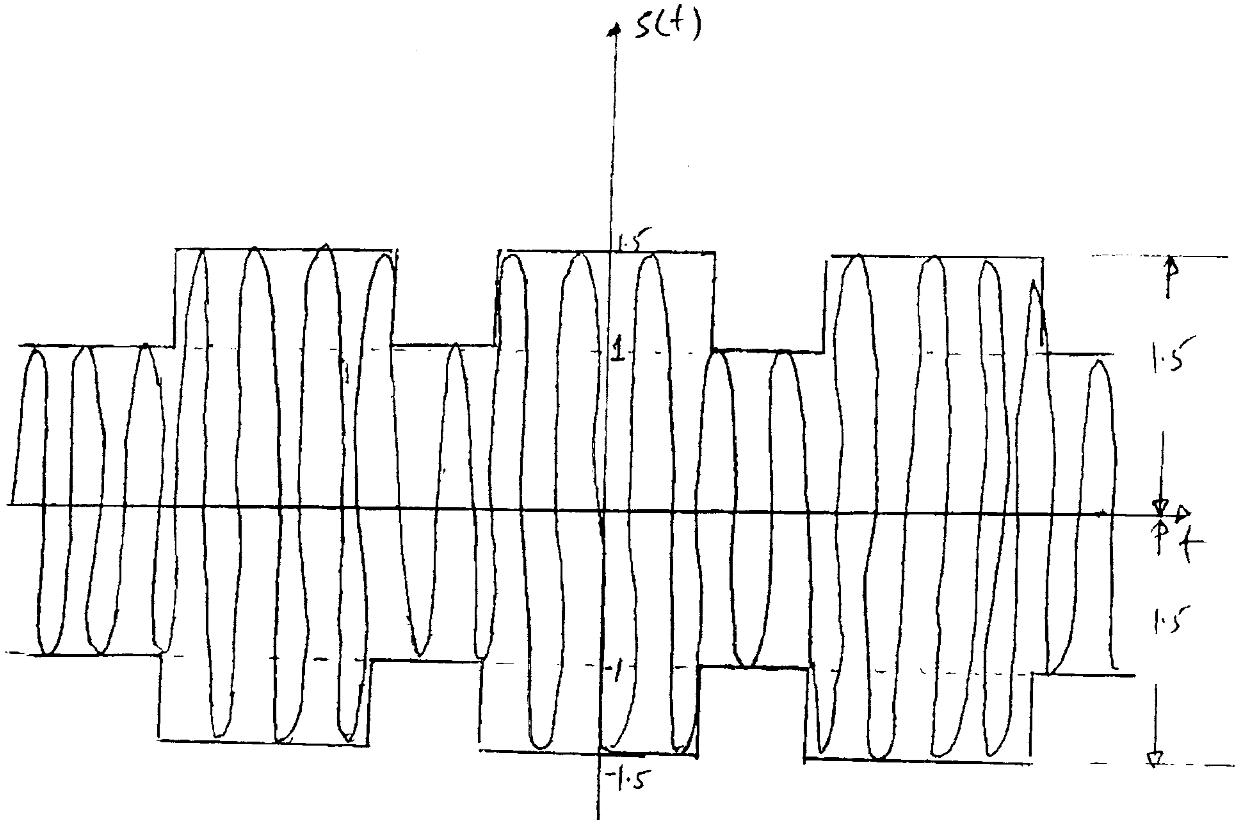
$$\text{MAX} | k_a m(t) | \times 100 = 50$$

$$\therefore k_a \cdot 1 = 50/100 = 1/2$$

$$s(t) = [1 + \tfrac{1}{2} m(t)] \cos(2\pi f_c t)$$

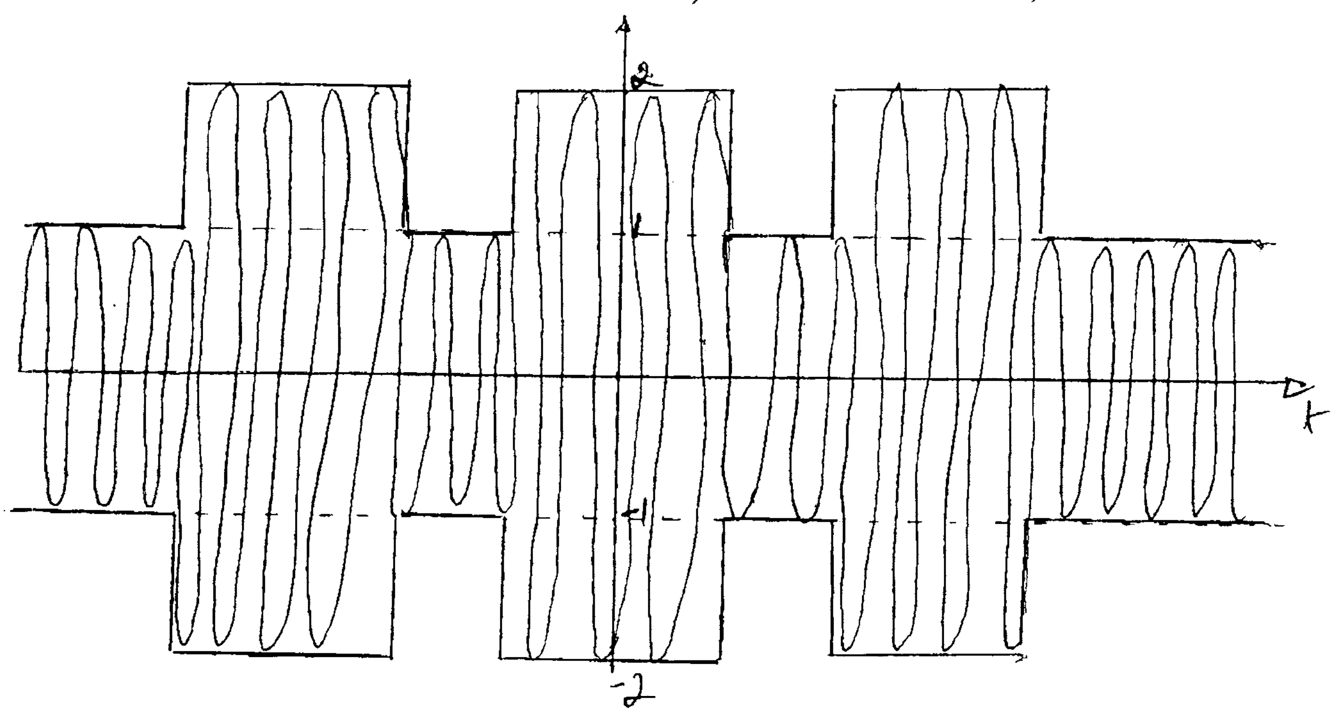
$$\max |s(t)| = \max [|1 + \tfrac{1}{2} m|] = 1.5$$

$$\min |s(t)| = \min [|1 + \tfrac{1}{2} \cdot m|] = 1$$



(2) 100% MODULATION: $k_a \cdot 1 \cdot 100 = 100$

$k_a = 1$, $\max|s(t)| = 2$, $\min|s(t)| = 1$



(b) THE MESSAGE SIGNAL $m(t)$ IS A SQUARE WAVE WHICH CAN BE REPRESENTED AS A FOURIER SERIES,

$$m(t) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{+j2\pi nt/T_0} \longrightarrow (3.2)$$

FOR THE SQUARE WAVE, c_n IS GIVEN BY

$$c_n = \tfrac{1}{2} \cdot \operatorname{sinc}(n/2)$$

$c_0 = 1/2$, $c_1 = 0.3183$, $c_3 = -0.1060$

EVEN HARMONICS ARE ZERO.

SINCE

$$1 \rightleftharpoons \delta(f)$$

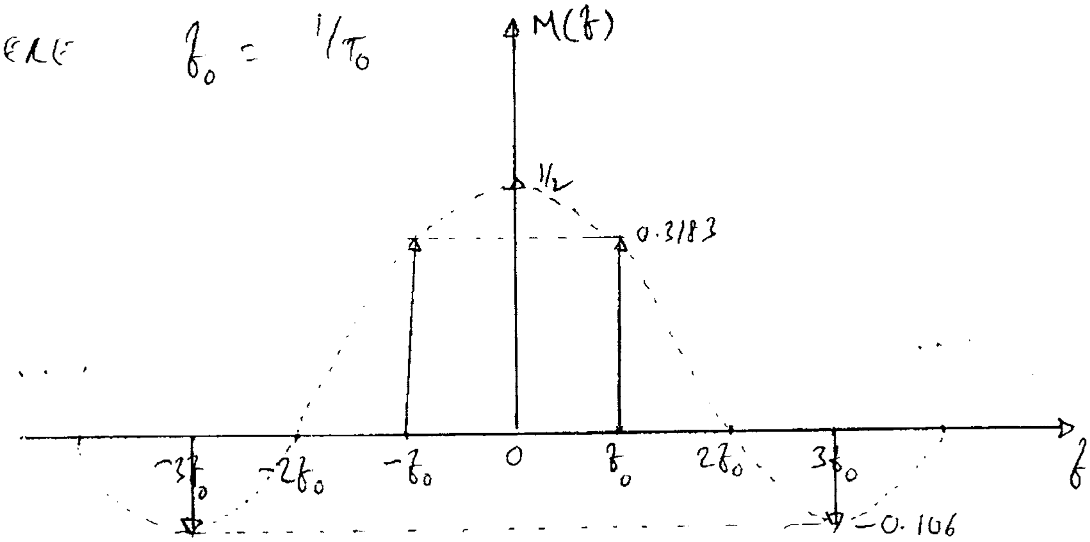
FROM FREQUENCY SHIFTING PROPERTY, IT FOLLOWS THAT

$$c_n \cdot e^{j2\pi nt/T_0} \rightleftharpoons c_n \cdot \delta(f - n/T_0)$$

FOURIER TRANSFORM OF $m(t)$ CAN BE WRITTEN AS

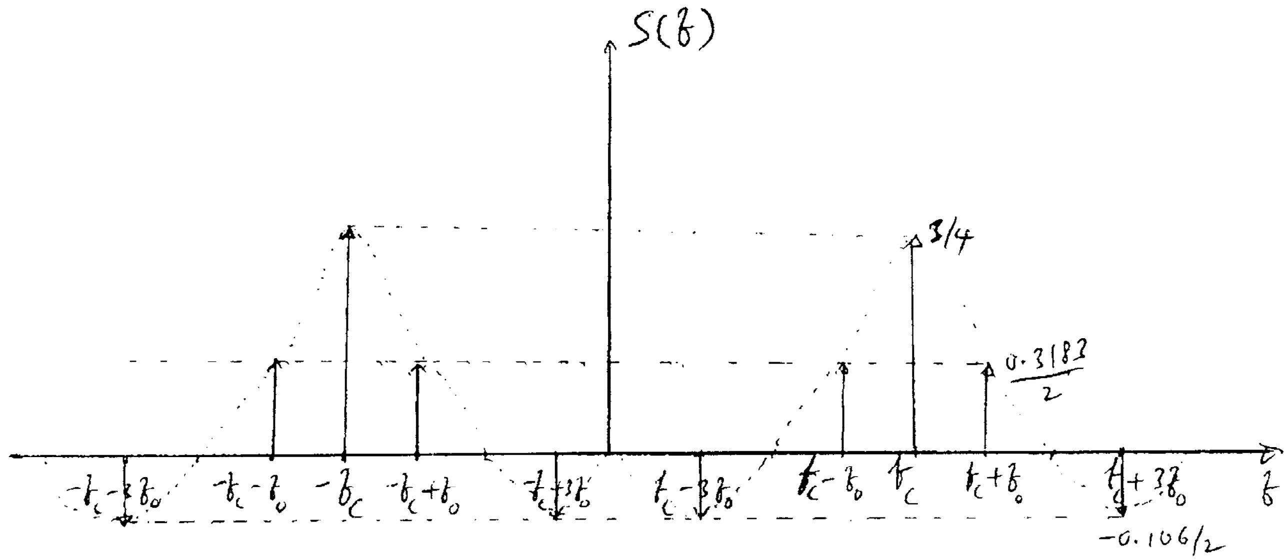
$$M(f) = \sum_{n=-\infty}^{\infty} c_n \cdot \delta(f - n f_0)$$

WHERE $f_0 = 1/T_0$



THE SPECTRUM OF THE AM WAVE IS

$$S(f) = \frac{1}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right] + \frac{1}{2}\left[M(f - f_c) + M(f + f_c)\right]$$


S(f)
3/4
0.3183/2
-fc-3fo
-fc-fo
-fc
-fc+fo
-fc+3fo
fc-3fo
fc-fo
fc
fc+fo
fc+3fo
f
-0.106/2