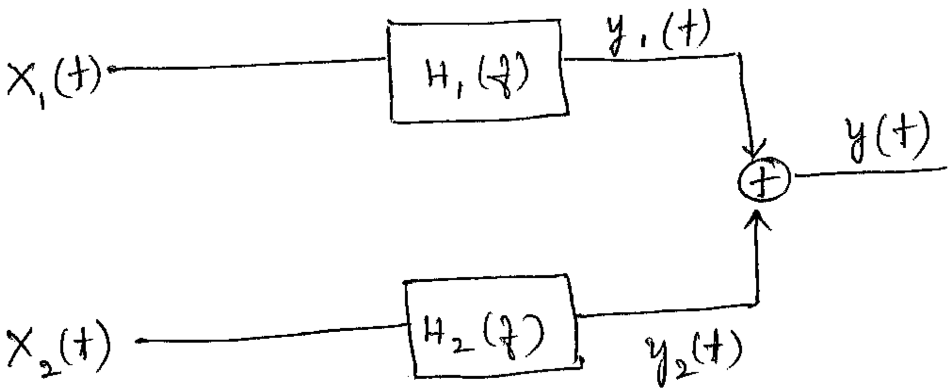
PRACTICE PROBLEMS

RANDOM PROCESSES

1, The inputs $X_1(t)$ & $X_2(t)$ are uncorrelated white noise processes each with power spectral density $N_0/2$ and zero-mean. The filter $H_1(f)$ is an ideal low pass filter with DC gain equal to 1 and cut off frequency f_0. The filter $H_2(f)$ is given by,

$$H_2(f) = \frac{1}{1 + j2\pi(f/a)}, \quad a > 0$$

Find the auto correlation $R_y(\tau)$ and power spectral density, $S_y(f)$ corresponding to the output $y(t)$.



Let $y_1(t)$ & $y_2(t)$ be the outputs from the filters $H_1(f)$ & $H_2(f)$, respectively.

$$\therefore \; y(t) = y_1(t) + y_2(t) \longrightarrow (1)$$

$$S_{y_1}(f) = |H_1(f)|^2 \cdot S_{x_1}(f)$$

$$S_{x_1}(f) = N_0/2, \quad |H_1(f)|^2 = \text{rect}(f/2f_0)$$

$$S_{y_1}(f) = \frac{N_0}{2} \cdot \text{rect}(f/2f_0)$$

$$R_{y_1}(t) = N_0/2 \cdot 2f_0 \cdot \text{sinc}(2f_0 t)$$

$$S_{y_2}(f) = |H_2(f)|^2 \cdot S_{x_2}(f)$$

$$= \frac{N_0}{2} \cdot \left| \frac{1}{1+j2\pi f/a} \right|^2$$

$$= \frac{N_0}{2} \left(\frac{1}{1+j2\pi f/a} \right) \cdot \left(\frac{1}{1-j2\pi f/a} \right) \quad \left(\because |A|^2 = A \cdot A^* \right)$$

$$= \frac{N_0}{2} \cdot \left[\frac{1}{1+\left(\frac{2\pi f}{a}\right)^2} \right] = \frac{N_0}{2} \left[\frac{a^2}{a^2+(2\pi f)^2} \right]$$

Since $e^{-a|t|} \rightleftharpoons \dfrac{2a}{a^2+(2\pi f)^2}$

The autocorrelation $R_{y_2}(t) = \left(\frac{N_0}{2}\right) \cdot \frac{a}{2} \cdot e^{-a|t|}$

$\therefore$ From 1, it follows that

$$R_y(t) = R_{y_1}(t) + R_{y_2}(t)$$

$$= N_0 f_0 \cdot \text{sinc}(2f_0 t) + \frac{N_0 a}{4} \cdot e^{-a|t|}$$

and

$$S_y(f) = S_{y_1}(f) + S_{y_2}(f)$$
$$= \frac{N_0}{2}\left[rect(f/2f_0) + \frac{a^2}{a^2 + (2\pi f)^2} \right]$$

2) Find the autocorrelation and power spectral density of a noise process $Y(t)$ given by

$$Y(t) = X(t) + \sin(2\pi f_c t + \Theta)$$

where $X(t)$ is a white noise process with zero mean, autocorrelation $R_X(\tau)$ & power spectral density $S_X(f)$. Θ is a random variable with uniform distribution.

$$f(\theta) = \begin{cases} 1/2\pi & \text{for } 0 < \theta < 2\pi \\ 0 & \text{elsewhere} \end{cases}$$

Assume Θ and $X(t)$ are independent.

$$R_y(\tau) = E[y(t+\tau) \cdot y(t)]$$
$$= E\left[\{X(t+\tau) + \sin[2\pi f_c(t+\tau) + \Theta]\} \{X(t) + \sin(2\pi f_c t + \Theta)\} \right]$$

$$= E\left[X(t+\tau)\cdot x(t)\right] + \underbrace{E\left[X(t+\tau)\cdot\sin(2\pi f_c t + \Theta)\right]}_{0}$$

$$+ \underbrace{E\left[\sin(2\pi f_c (t+\tau) + \Theta)\, X(t)\right]}_{0}$$

$$+ E\left[\sin(2\pi f_c (t+\tau) + \Theta)\, \sin(2\pi f_c t + \Theta)\right]$$

The second & third terms vanish because Θ & $X(t)$ are independent,

i.e. $E\left[X(t+\tau).\ \sin(2\pi f_c t + \Theta)\right]$

$$= \underbrace{E\left[x(t+\tau)\right]}_{=0} \cdot E\left[\sin(2\pi f_c t + \Theta)\right]$$

using $\sin A \cdot \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$$E\left[\sin(2\pi f_c (t+\tau) + \Theta))\cdot \sin(2\pi f_c t + \Theta)\right]$$

$$= \frac{1}{2}\left\{E\left[\overset{\cos}{\cancel{\sin}}(2\pi f_c \tau)\right] - \underbrace{E\left[\overset{\cos}{\cancel{\sin}}(4\pi f_c t + 2\pi f_c \tau + 2\Theta)\right]}_{=0}\right.$$

$$= \frac{1}{2}\cdot \overset{\cos}{\cancel{\sin}}(2\pi f_c \tau)$$

$$\therefore\ R_y(\tau) = R_X(\tau) + \frac{1}{2}\cdot \overset{\cos}{\cancel{\sin}}(2\pi f_c \tau)$$

3) $X(t)$ is a stationary random process with the auto correlation function given by,

$$R_X(\tau) = e^{-a|\tau|}.$$

The input random process passes through a filter whose transfer function is given by

$$H(f) = \frac{1}{1+j2\pi f/b}, \quad b>0$$

Find the output auto-correlation function $R_Y(\tau)$ & power spectral density $S_Y(f)$. Assume $b \neq a$.

Input PSD, $S_X(f) = F[R_X(\tau)]$.

$$= \frac{2a}{a^2+(2\pi f)^2}$$

$$|H(f)|^2 = \left(\frac{1}{1+j2\pi f/b}\right) \cdot \frac{1}{\left(1-j2\pi f/b\right)} = \frac{1}{1+\left(\frac{2\pi f}{b}\right)^2}$$

$$\therefore S_Y(f) = S_X(f)\,|H(f)|^2$$

$$= \frac{2a}{a^2+(2\pi f)^2} \cdot \frac{b^2}{b^2+(2\pi f)^2}$$

Let $(2\pi f)^2 = u$. Now the right hand side can be written as

$$RHS = \frac{2a}{a^2+u} \cdot \frac{b^2}{b^2+u}$$

$$= \frac{2ab^2}{(b^2-a^2)} \left\{ \frac{1}{a^2+u} - \frac{1}{b^2+u} \right\}$$

$$= \frac{2ab^2}{(b^2-a^2)} \left\{ \frac{1}{a^2+(2\pi f)^2} - \frac{1}{b^2+(2\pi f)^2} \right\}$$

The output autocorrelation function $R_y(\tau)$ is given by.

$$R_y(\tau) = \frac{2ab^2}{(b^2-a^2)} \cdot \left\{ \frac{1}{2a} \cdot e^{-a|\tau|} - \frac{1}{2b} \cdot e^{-b|\tau|} \right\}$$