EXAMPLE: SINUSOIDAL WAVE WITH RANDOM PHASE

CONSIDER THE RANDOM PROCESS DENOTED BY

$$X(t) = A\cos(2\pi f_c t + \Theta) \longrightarrow (25)$$

WHERE A & f_c ARE CONSTANTS. Θ IS A RANDOM VARIABLE AND ~~ITS~~ CAN TAKE ANY VALUE BETWEEN 0 & 2π WITH EQUAL PROBABILITY, i.e. ITS pdf IS ~~UNIFORM~~ GIVEN BY

$$f(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \le \theta < 2\pi \\ 0 & \text{ELSEWHERE} \end{cases} \longrightarrow (26)$$

FIND THE MEAN, VARIANCE & AND AUTO-CORRELATION.

THE MEAN IS GIVEN BY

$$E[X(t)] = E[A\cos(2\pi f_c t + \theta)]$$

$$= \cancel{A\cos(2\pi f_c t) E[}$$

$$= E[A\cdot\cos(2\pi f_c t)\cdot\cos\theta - A\sin(2\pi f_c t)\cdot\sin\theta]$$

$$= A\cos(2\pi f_c t)\, E[\cos\theta] - A\sin(2\pi f_c t)\, E(\sin\theta) \longrightarrow (27)$$

BY DEFINITION, $E[\cos(\theta)] = \cancel{\int_{-\infty}^{\infty} f(\theta)\cos\theta\, d\theta} = \int_{-\infty}^{\infty} f(\theta)\cos\theta\cdot d\theta$

i.e. $$E[\cos\theta] = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta \, d\theta = 0 \longrightarrow (28)$$

SIMILARLY $$E(\sin\theta) = 0 \longrightarrow (29)$$

USING (28) & (29) IN (27), WE FIND THAT

$$E[X(t)] = 0 \longrightarrow (30)$$

SINCE THE MEAN IS ZERO, VARIANCE IS SIMPLY THE SECOND MOMENT, i.e.

$$\sigma_X^2 = E[X^2(t)] = E[A^2 \cos^2(2\pi f_c t + \theta)]$$

$$= E\left\{\frac{A^2}{2}[1 + \cos(4\pi f_c t + 2\theta)]\right\}$$

$$= \frac{A^2}{2} + \frac{A^2}{2} E[\cos(4\pi f_c t + 2\theta)]$$

$$= \frac{A^2}{2} + \frac{A^2}{2} E[\cos(4\pi f_c t)\cos(2\theta) - \sin(4\pi f_c t)\sin(2\theta)]$$

$$= \frac{A^2}{2}$$ SINCE $E(\cos 2\theta)$ & $E(\sin 2\theta)$ ARE ZEROS SIMILAR TO EQ. (28) & (29).

THE AUTO-CORRELATION FUNCTION OF $X(t)$ IS

$$R_X(\tau) = E\left[X(t+\tau)X(t)\right]$$

$$= E\left\{\cos\left[2\pi f_c(t+\tau)+\theta\right]\cdot\cos\left[2\pi f_c t+\theta\right]\right\}\cdot A^2$$

$$= \frac{A^2}{2} E\left\{\cos\left[4\pi f_c t + 2\pi f_c \tau + 2\theta\right] + \cos\left[2\pi f_c \tau\right]\right\}$$

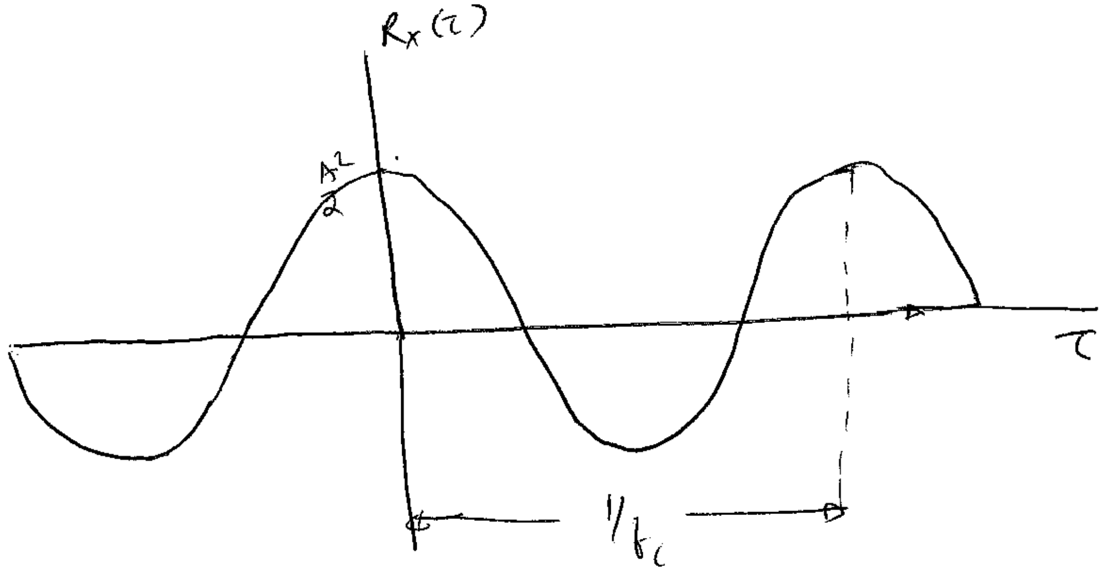
$$= \frac{A^2}{2}\left\{E\left[\cos(\alpha+2\theta)\right] + E\left[\cos(2\pi f_c \tau)\right]\right\}$$

WHERE $\alpha = 4\pi f_c t + 2\pi f_c \tau$;

CONSIDER $E\left[\cos(\alpha+2\theta)\right] = E\left[\cos\alpha\cdot\cos(2\theta) - \sin\alpha\cdot\sin(2\theta)\right]$

$$= 0$$

$\therefore R_X(t) = \frac{A^2}{2}\cos(2\pi f_c \tau)$ ($\because \cos(2\pi f_c \tau)$ IS DETERMINISTIC)



FREQ. DOMAIN REPRESENTATION OF NOISE

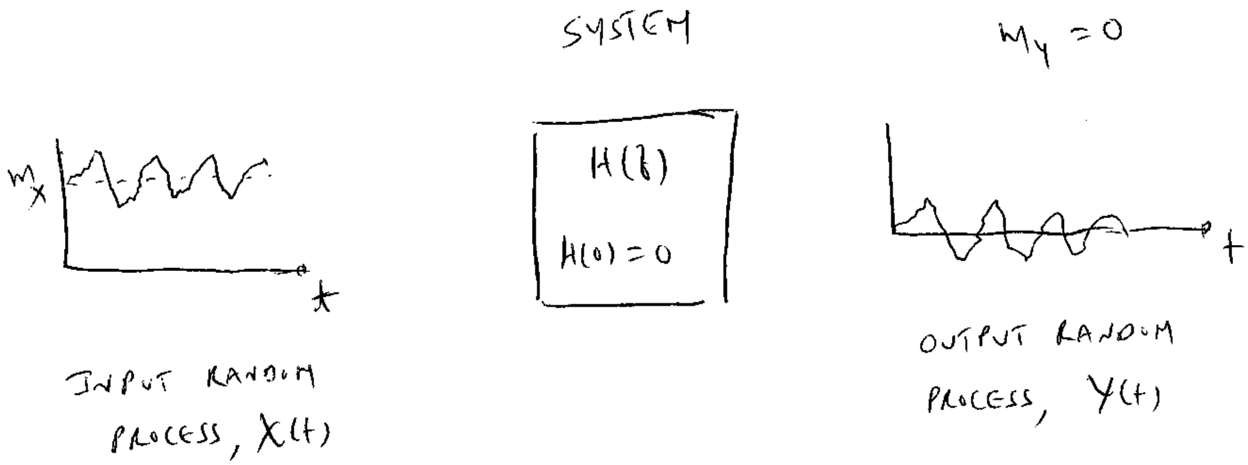
WHEN A SIGNAL CORRUPTED BY NOISE PASSES THROUGH A FILTER, FREQUENCY COMPONENTS OF THE INPUT SIGNAL (i.e, SIGNAL + NOISE) IS ALTERED BY THE FILTER. TO UNDERSTAND THE IMPACT OF FILTER, IT IS NECESSARY TO KNOW THE SPECTRAL CHARACTERISTICS OF THE NOISE. FOR EXAMPLE, WE LIKE TO KNOW WHAT HAPPENS WHEN A RANDOM PROCESS WITH MEAN m_X PASSES THROUGH A FILTER. SINCE THE MEAN m_X IS CONSTANT FOR A STATIONARY RANDOM PROCESS, ITS FOURIER TRANSFORM IS A DELTA FUNCTION, i.e,

$$F\{E(x(t))\} = F[m_X] = m_X \cdot \delta(f)$$

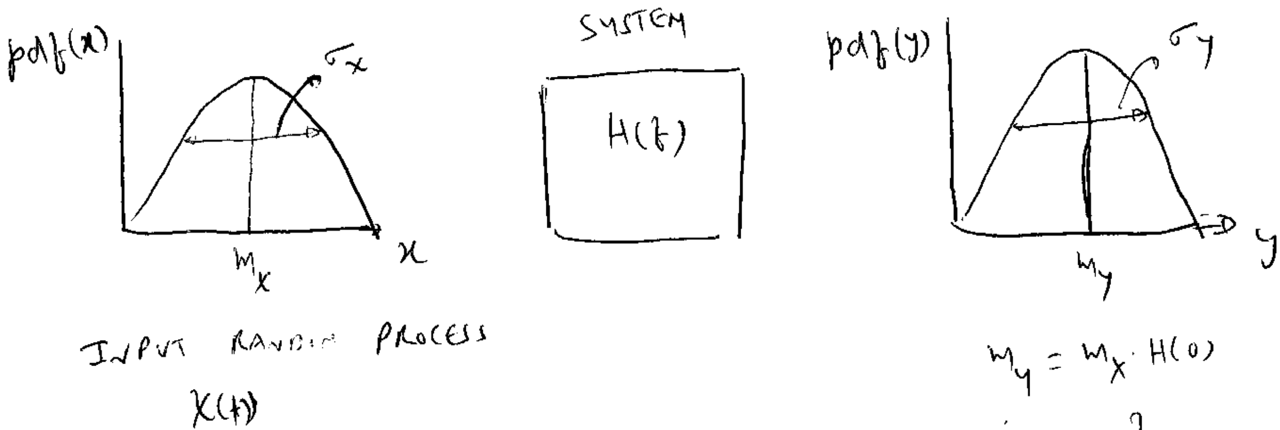
THE MEAN OF THE INPUT SIGNAL IS MULTIPLIED BY THE FILTER TRANSFER FUNCTION $H(f)$ AT $f=0$, i.e., THE MEAN OF THE OUTPUT SIGNAL, $y(t)$ IS GIVEN BY

$$m_y = m_X \cdot H(0) \qquad (31)$$

FOR EXAMPLE, IF THE INPUT RANDOM SIGNAL PASSES THROUGH A CAPACITOR, THE D.C. COMPONENT IS BLOCKED AND THEREFORE, THE MEAN OF THE OUTPUT RANDOM PROCESS $y(t)$ IS ZERO, AS SHOWN IN THE FIGURE.



SUPPOSE THE VARIANCE OF THE INPUT RANDOM PROCESS IS σ_X^2. WHAT WOULD BE THE ~~OUTPUT~~ VARIANCE OF THE OUTPUT RANDOM PROCESS ?



$m_y = m_X \cdot H(0)$

$\sigma_y = ?$

SINCE VARIANCE IS A SPECIAL CASE OF AUTO-CORRELATION WITH A TIME SHIFT OF ZERO, WE CAN AS WELL INVESTIGATE THE FREQ. DOMAIN REPRESENTATION OF AUTO-CORRELATION FUNCTION GIVEN BY,

$$R_X(t) = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(\tau)\, x(\tau + t)\, d\tau \longrightarrow (32)$$

INSTEAD OF THE MEASUREMENT INTERVAL $[0, T]$, WE CHOOSE A SYMMETRIC INTERVAL $[-T/2, T/2]$. THE VALUE OF THE CORRELATION FUNCTION IS ~~INVARIANT~~ UNAFFECTED BY THIS SHIFT.

$$R_X(t) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(\tau)\, x(t + \tau)\, d\tau \longrightarrow (33)$$

SINCE $R_X(t)$ IS SYMMETRIC, $R_X(t) = R_X(-t)$, EQ. (33) MAY BE REWRITTEN AS

$$R_X(t) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(\tau)\, x(\tau - t)\, d\tau \longrightarrow (34)$$

EQ. (34) LOOKS SIMILAR TO CONVOLUTION OF $x(t)$ WITH ITSELF. BUT IN THE CASE OF CONVOLUTION, ~~THE~~ ONE OF THE SIGNALS IS INVERTED BEFORE THE TIME SHIFT, i.e.

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t - \tau)\, d\tau \longrightarrow (35)$$

IN EQ. (35), IF WE CHOOSE $x_1(t) = x(t)$ & ~~$x_2(t) = x(-t)$~~ $x_2(t) = x(-t)$, $x_2(t-\tau)$ ON THE RHS BECOMES ~~x_2~~ $x(t-\tau)$, i.e.,

$$\cancel{x_1(t) * x_2(t)} \quad x(t) * x(-t) = \int_{-\infty}^{\infty} x(\tau)\, x(\tau - t)\, d\tau \quad \longrightarrow (36)$$

THE RHS OF EQ. (36) IS THE SAME AS THAT OF THE INTEGRAL OF EQ. (34).
SO, WE WRITE

$$R_X(t) = \lim_{T \to \infty} \frac{1}{T} \left[x(t) * x(-t) \right] \longrightarrow (37)$$

USING THE CONVOLUTION THEOREM, CONVOLUTION IN TIME DOMAIN MAY BE WRITTEN AS MULTIPLICATION IN FREQUENCY DOMAIN, i.e.

$$F\left[R_X(t)\right] = \lim_{T \to \infty} \frac{1}{T} \, X(f) \cdot F\left[x(-t)\right] \longrightarrow (38)$$

WHERE $X(f) = F\left[x(t)\right]$.

(28)

SINCE $F[x(-t)] = X^*(f)$ †, IT FOLLOWS THAT

$$F[R_X(t)] = \lim_{T \to \infty} \frac{1}{T} |X(f)|^2 \longrightarrow (39)$$

THE QUANTITY ON THE RHS OF EQ. (39) IS CALLED THE POWER SPECTRAL DENSITY (PSD). THEREFORE, PSD AND AUTO-CORRELATION FUNCTION ARE FOURIER TRANSFORM PAIRS.

$$PSD = \lim_{T \to \infty} \frac{1}{T} |X(f)|^2 \longrightarrow (40)$$

$$R_X(t) \rightleftharpoons PSD(f) \longrightarrow (41)$$

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† ~~If~~ $x(t) \rightleftharpoons X(f)$, T~~hen~~ CONSIDER

$$\cancel{x(-t) = \int_{-\infty}} \quad F[x(-t)] = \int_{-\infty}^{\infty} x(-t)\, e^{-j2\pi f t}\, dt.$$

Let $t = -u$, $dt =$

$$F[x(-t)] = \int_{-\infty}^{\infty} x(u)\, e^{j2\pi f u}\, du = X(-f)$$

FOR REAL SIGNALS, ~~it~~ WE HAVE $X(-f) = X^*(f)$