PROPERTIES OF THE POWER SPECTRAL DENSITY:

THE PSD & AUTOCORRELATION ARE A FOURIER TRANSFORM PAIR:

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \cdot e^{-j2\pi f\tau} d\tau \longrightarrow (52)$$

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) \cdot e^{j2\pi f\tau} df \longrightarrow (53)$$

PROPERTY 1:

DC VALUE OF THE PSD IS EQUAL TO THE AREA UNDER THE AUTOCORRELATION FUNCTION, i.e.,

$$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau) d\tau \longrightarrow (54)$$

THIS FOLLOWS FROM (52) BY SETTING $f = 0$.

PROPERTY 2:

THE MEAN SQUARED VALUE OF A STATIONARY PROCESS IS THE AREA UNDER THE GRAPH OF THE PSD.

$$E[X^2(t)] = R_X(0) = \int_{-\infty}^{\infty} S_X(f) df \longrightarrow (55)$$

IF THE MEAN ~~...~~ $m_X = 0$, $E[X^2(t)] = \sigma_X^2$, VARIANCE.

$$\therefore \quad \sigma_X^2 = R_X(0) = \int_{-\infty}^{\infty} S_X(f)\,df$$

THIS FOLLOWS BY SETTING $\tau = 0$ IN (53).

EXAMPLE: CONSIDER A WHITE NOISE PROCESS $X(t)$ WITH PSD $S_X(f) = A^2$ PASSING THROUGH AN IDEAL LOW PASS FILTER. THE PSD OF THE OUTPUT NOISE PROCESS $Y(t)$ IS

$$S_Y(f) = S_X(f)\,|H(f)|^2 = A^2 |H(f)|^2$$
$$= A^2 \,\mathrm{rect}(f/2f_c)$$

WHERE f_c IS THE CUTOFF FREQUENCY OR BANDWIDTH OF THE FILTER.

$$\sigma_Y^2 = \int_{-\infty}^{\infty} S_Y(f)\,df \quad \longrightarrow (56)$$

$$= A^2 \int_{-f_c}^{f_c} df = 2 f_c A^2 \quad \longrightarrow (57)$$

OR

$$\therefore \quad \boxed{\text{VARIANCE } \propto \text{ BANDWIDTH}}$$

NOTE THAT THE VARIANCE OF $X(t)$ IS INFINITE AND THE VARIANCE OF THE OUTPUT RANDOM PROCESS DECREASES AS $f_c \downarrow$. THIS IS BECAUSE A FILTER WITH NARROW PASS BAND TRUNCATES MOST OF THE FREQUENCY COMPONENTS AND THEREFORE, THE OUTPUT NOISE VARIANCE IS DIRECTLY PROPORTIONAL TO FILTER BANDWIDTH.

ALSO NOTE THAT σ_Y^2 IS THE AVERAGE NOISE POWER (Eq. (87)). THE RHS OF Eq. (56) IS THE SUM OF THE LOCAL POWER SPECTRAL DENSITY MULTIPLIED BY THE FREQUENCY INTERVAL Δf. THIS SUM IS EQUAL TO THE AVERAGE NOISE POWER, AS IT SHOULD BE.

<u>PROPERTY 3</u>:

$$S_X(f) \geq 0 \quad \forall f \qquad \rightarrow (58)$$

$S_X(f)$ IS PURE REAL

THIS PROPERTY FOLLOWS FROM THE DEFINITION

$$S_X(f) = \lim_{T \to \infty} \frac{1}{T} |X(f)|^2$$

PROPERTY 4:

THE PSD OF A REAL RANDOM PROCESS IS AN EVEN FUNCTION

$$S_X(-f) = S_X(f) \longrightarrow 59$$

PROOF:

$$F[R_X(t)] = S_X(f) = \int_{-\infty}^{\infty} R_X(t) \cdot e^{-j2\pi f t} \, dt$$

$$S_X(-f) = \int_{-\infty}^{\infty} R_X(t) \cdot e^{+j2\pi f t} \cdot dt$$

SINCE $R_X(t) = R_X(-t)$

$$S_X(-f) = \int_{-\infty}^{\infty} R_X(-t) \cdot e^{-j2\pi f(-t)} \, dt$$

CHANGING THE VARIABLE OF INTEGRATION: $-t = u$

$$S_X(-f) = \int_{-\infty}^{\infty} R_X(u) \cdot e^{-j2\pi f u} \, du$$

$$= S_X(f)$$

(FOURIER TRANSFORM OF A SYMMETRIC FUNCTION IS SYMMETRIC)

~~(FOURIER TRANSFORM OF AN EVEN FUNCTION IS EVEN~~)

EXAMPLE: FIND THE AUTO CORRELATION $R_Y(\tau)$ AND PSD $R_Y(f)$ OF THE OUTPUT $Y(t)$ FROM THE FOLLOWING FILTER

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

THE IMPULSE RESPONSE FUNCTION $h(t) = e^{-at}\, u(t), \quad a > 0$

WHERE

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{ELSEWHERE} \end{cases}$$

THE PROCESS $X(t)$ IS WHITE, WITH PSD $S_X(f) = K$.

THE OUTPUT PSD $\cancel{R_Y(f) = R_X(f)}$

$$S_Y(f) = S_X(f)\, |H(f)|^2$$

$$= K\, |H(f)|^2 \longrightarrow (60)$$

$$H(f) = \int_{-\infty}^{\infty} h(t) \cdot e^{-j2\pi f t}\, dt = \int_{0}^{\infty} e^{-[a+j2\pi f]t} \cdot dt$$

$$= \left. \frac{-1}{a+j2\pi f} \right|_{0}^{\infty} = \frac{1}{a+j2\pi f}$$

$$H(f) = \left. \frac{e^{-(a+j2\pi f)t}}{-(a+j2\pi f)} \right|_0^\infty = \frac{1}{a+j2\pi f}$$

$$H^*(f) = \frac{1}{a-j2\pi f}$$

$$|H(f)|^2 = H(f)\cdot H^*(f) = \frac{1}{a^2+4\pi^2 f^2} \rightarrow (61)$$

$$\therefore S_y(f) = K\,|H(f)|^2 = \frac{K}{a^2+4\pi^2 f^2} \rightarrow (62)$$

THE AUTOCORRELATION OF THE OUTPUT $R_y(\tau) = F^{-1}[S_y(f)]$.

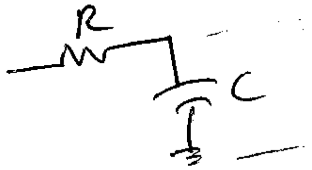
NOTE THAT (HAYKIN, P. 764)

$$e^{-a|t|},\ a>0 \rightleftharpoons \frac{2a}{a^2+(2\pi f)^2} \rightarrow (63)$$

$$\therefore R_y(\tau) = F^{-1}[S_y(f)] = \frac{K}{2a}\, F^{-1}\left[\frac{2a}{a^2+(2\pi f)^2}\right]$$

i.e. $$R_y(\tau) = \frac{K}{2a} \cdot e^{-a|\tau|} \longrightarrow (64)$$

NOTE THAT EQ (61) CORRESPONDS TO THE MAGNITUDE SQUARED FREQUENCY RESPONSE OF A FIRST ORDER LOW PASS FILTER (WITHIN A SCALE FACTOR).



FREQ. RESPONSE $$H(f) = \frac{1}{1 + j2\pi f RC}$$

Taking $RC = 1/a$,

$$|H(f)|^2 = \frac{a^2}{a^2 + (2\pi f)^2}.$$