TUTORIAL QUESTIONS: RANDOM VARIABLES & PROCESSES

1. CONSIDER THE RANDOM VARIABLE X WHICH HAS THE FOLLOWING DISTRIBUTION:

$$f(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) FIND THE MEAN AND VARIANCE OF X

(b) FIND THE EXPECTATION OF x^3.

SOLUTION:

$$\text{MEAN } m_X = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_0^1 x \cdot 2x\, dx = 2 \cdot \frac{x^3}{3} \Big|_0^1$$

$$= 2/3 .$$

$$\text{VARIANCE } \sigma_X^2 = E\left[(X - m_x)^2\right] = \int_{-\infty}^{\infty} (x - 2/3)^2 f(x)\, dx$$

$$= \int_0^1 (x - 2/3)^2\, 2x\, dx$$

$$\sigma_x^2 = 2\int_0^1 \left[x^2 + \frac{4}{9} - \frac{4x}{3} \right] x \, dx$$

$$= 2 \int_0^1 \left[x^3 + \frac{4x}{9} - \frac{4x^2}{3} \right] dx$$

$$= 2 \left\{ \frac{x^4}{4} + \frac{\overset{2}{\cancel{4}} x^2}{9 \cdot \cancel{2}} - \frac{4x^3}{3 \cdot 3} \Big|_0^1 \right\}$$

$$= 2 \cdot \left\{ \frac{1}{4} + \frac{2}{9} - \frac{4}{9} \right\} = 2 \left\{ \frac{9-8}{36} \right\}$$

$$= 1/18$$

$$E[X^3] = \int_{-\infty}^{\infty} x^3 f(x) \, dx = \int_0^1 x^3 \cdot 2x \, dx$$

$$= 2 \cdot \frac{x^5}{5} \Big|_0^1 = 2/5$$

2. THE RANDOM VARIABLE X HAS THE FOLLOWING GAUSSIAN DISTRIBUTION:

$$f(x) = \frac{1}{\left(2\pi \cdot 1/2\right)^2} \exp\left[\frac{-1}{2 \cdot 1/2}(x-3)^2\right]$$

FIND THE MEAN AND VARIANCE

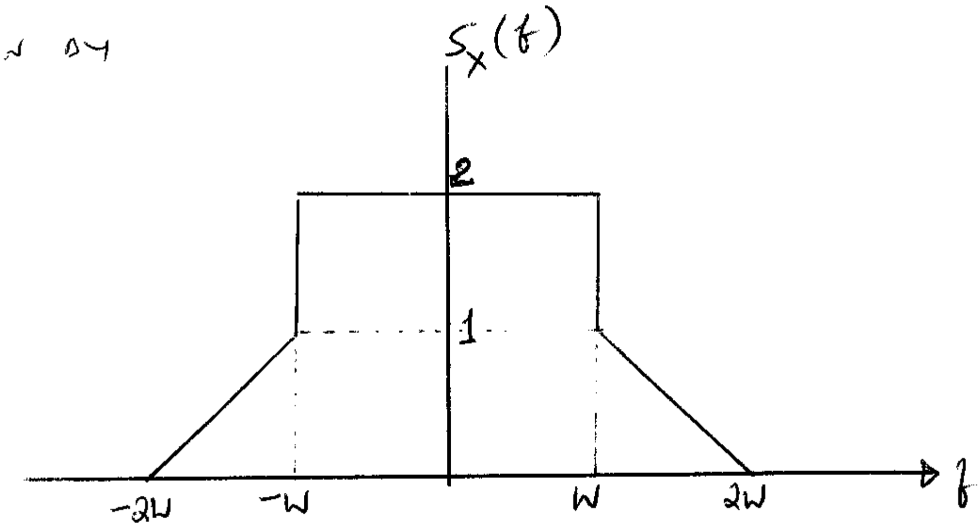
SOLUTION: THE STANDARD GAUSSIAN DISTRIBUTION WITH MEAN m_X AND VARIANCE σ_X^2 IS

$$f(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left[-\frac{1}{2\sigma_X^2}(x-m_X)^2\right]$$

COMPARING THE ABOVE WITH THE PDF OF THE QUESTION, WE HAVE

$$\sigma^2 = 1/2 \qquad m_X = 3$$

3. FIND THE POWER IN THE SIGNAL WHOSE PSD $S_X(f)$ IS GIVEN BY



FROM THE PROPERTIES OF PSD, AREA UNDER THE PSD IS THE POWER IN THE SIGNAL ∴

$$\sigma_X^2 = \text{POWER} = \int_{-\infty}^{\infty} S_X(f)\,df$$

THE AREA OF THE RECTANGLE (FOR POSITIVE FREQ.) = $2W$.

THE AREA OF THE TRIANGLE (FOR POSITIVE FREQ) = $W/2$

∴ TOTAL AREA (BECAUSE OF SYMMETRY) = $\left[2W + W/2\right]2$

$= 5W$.

∴ POWER = $\sigma_X^2 = 5W$.

NOTE THIS IS ALSO EQUAL TO $R_X(0)$.

5

4 Kumar

4. A random process $X(t)$ is multiplied by a sinusoidal wave $\cos(2\pi f_c t + \theta)$ where the phase θ is a random variable that is uniformly distributed over $(0, 2\pi)$. The autocorrelation function and PSD of $X(t)$ are $R_X(t)$ and $S_X(f)$. Find the autocorrelation and PSD of the random process defined by

$$Y(t) = X(t) \cos(2\pi f_c t + \theta)$$

Assume that the random process $X(t)$ & r.v. θ are independent.

Solution:

$$R_Y(\tau) = E[Y(t+\tau)Y(t)]$$

$$= E\Big[X(t+\tau)\cos(2\pi f_c t + 2\pi f_c \tau + \theta) * X(t)\cdot\cos(2\pi f_c t + \theta)\Big] \rightarrow (1)$$

Since the random process $X(t)$ & θ are independent, we can write (1) as

$$R_y(\tau) = E\left[X(t+\tau)\times X(t)\right] E\left[\cos(2\pi f_c t + 2\pi f_c \tau + \theta) \times \cos(2\pi f_c t + \theta)\right]$$

($\because$ IF A & B ARE INDEPENDENT R.V.s, $E[AB] = E[A] \cdot E[B]$)

$$R_y(\tau) = \frac{R_X(\tau)}{2} \cdot E\left[\cos(2\pi f_c \tau) + \cos(4\pi f_c t + 2\pi f_c \tau + 2\theta)\right]$$

($\because \cos(A) \cdot \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$)

$$R_y(\tau) = \frac{R_X(\tau)}{2} E\left[\cos(2\pi f_c \tau)\right] + \underbrace{E\left[\cos(4\pi f_c t + 2\pi f_c \tau + 2\theta)\right]}_{0} \longrightarrow (2)$$

THE FIRST TERM IN EQ. (2) IS DETERMINISTIC, THEREFORE

$$E\left[\cos(2\pi f_c \tau)\right] = \cos(2\pi f_c \tau)$$

THE SECOND TERM IS ZERO. (LECTURE NOTES ON RANDOM PROCESS, EXAMPLE CORRESPONDING TO EQ. 25).

$$\therefore R_y(\tau) = \frac{R_X(\tau)}{2} \cdot \cos(2\pi f_c \tau)$$

$$R_X(\tau) \rightleftharpoons S_X(f)$$

USING THE FREQUENCY SHIFTING PROPERTY OF THE FOURIER TRANSFORM, IT FOLLOWS THAT

$$S_y(f) = \frac{1}{4}\left[S_X(f - f_c) + S_X(f + f_c) \right]$$