CHAPTER 7

Modulation Techniques

Problem 1

(a) An AM wave is defined by

$$s(t) = [1 + k_a M(t)]c(t)$$

For the message signal

$$m(t) = 20\cos(2\pi t),$$

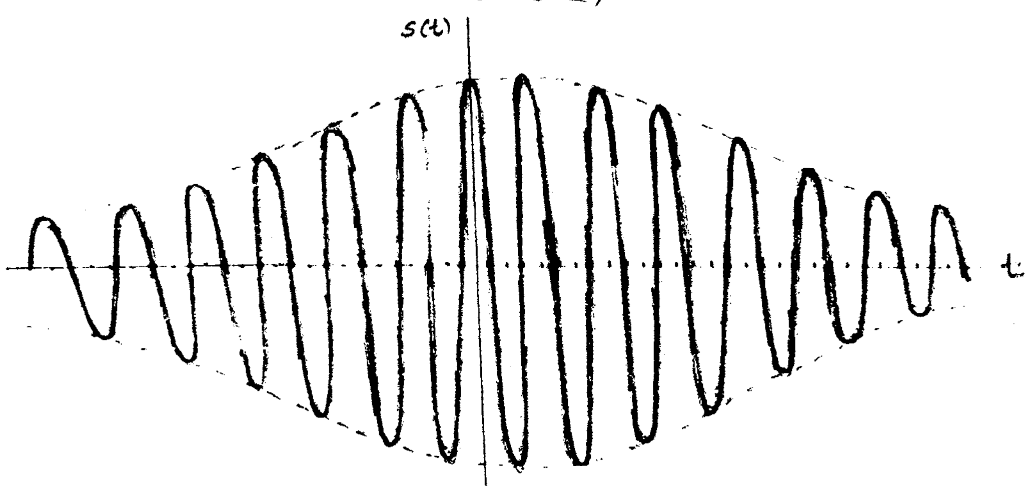
the carrier

$$c(t) = 50\cos(100\pi t),$$

and a percentage modulation of 75 percent, the corresponding AM wave is

$$s(t) = 50[1 + 0.75\cos(2\pi t)]\cos(100\pi t)$$

Hence, s(t) has the waveform (not to scale)



(b) The expression for s(t) may be expanded as

$$s(t) = 50\cos(100\pi t) + 37.5\cos(2\pi t)\cos(100\pi t)$$

$$= 50\cos(100\pi t) + 18.75\cos(102\pi t) + 7.5\cos(98\pi t)$$

Hence, the average power of s(t) is

$$P_{av} = \frac{1}{2}(50)^2 + \frac{1}{2}(18.75)^2 + \frac{1}{2}(18.75)^2$$

$$= 1250 + 351.56 = 1601.56$$

✓ Problem 2

The transfer function of the resonant circuit of Fig. P7.1, relating the output voltage v(t) to the input current i(t), is

$$H(f) = \frac{V(f)}{I(f)}$$

$$= \frac{1}{(1/R)+j(2\pi fC-1/2\pi fL)}$$

$$= \frac{R}{1+jQ(f-f_c)(f+f_c)/ff_c} \qquad (1)$$

where $V(f)= F[v(t)]$

$I(f)= F[i(t)]$

$$f_c = \frac{1}{2\pi\sqrt{LC}}$$

$$Q = R\sqrt{\frac{C}{L}}$$

When $Q>>1$, we find that the frequencies of interest lie in the neighbourhood of f_c, and so we may approximate Eq. (1) as follows:

$$H(f) \simeq \frac{R}{1+j2Q(f-f_c)/f_c}, \quad f > 0 \qquad (2)$$

The spectrum of the input AM wave i(t) consists of a carrier at f_c, and two side-frequencies at $f_c \pm f_m$, as shown by

$$i(t) = A_c \cos(2\pi f_c t)$$

$$+ \frac{A_c}{4}\{\cos[2\pi(f_c+f_m)t]+\cos[2\pi(f_c-f_m)t]\}$$

$$= A_c \cos(2\pi x\ 10^6 t)$$

$$+ 0.25\ A_c[\cos(2.01\pi x10^6)+\cos(1.99\pi x10^6 t)]$$

Therefore evaluating Eq.(2) at the frequencies f_c and $f_c \pm f_m$, we get

$$H(f_c) = R$$

$$H(f_c \pm f_m) \simeq \frac{R}{1\pm j2Qf_m/f_c}$$

$$= \frac{R}{1 \pm j2 \times 175 \times 5 \times 10^3/10^6}$$

$$= \frac{R}{1 \pm j1.75}$$

$$\simeq 0.5R \angle \mp 1.05$$

where the phase angles are expressed in radians. The resulting output AM wave is therefore defined by

$$v(t) \simeq A_c R \cos(2\pi \times 10^6 t)$$

$$+ 0.125\, A_c R[\cos(2.01\pi \times 10^6 t - 1.05) + \cos(1.99\pi \times 10^6 + 1.05)]$$

The percentage modulation of this modulated wave is 25 percent.

Problem 3

The AM wave is defined by

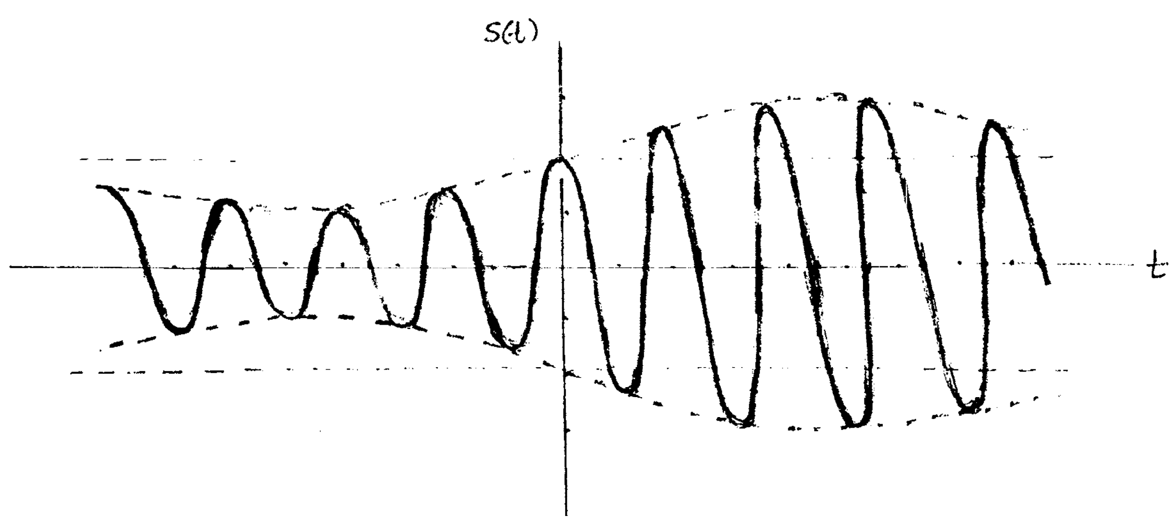
$$s(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)$$

$$= A_c(1 + \frac{k_a t}{1+t^2})\cos(2\pi f_c t)$$

(a) For 50% modulation, $k_a = 1$. Hence,

$$s(t) = A_c(1 + \frac{t}{1+t^2})\cos(2\pi f_c t),$$

which has the following waveform:



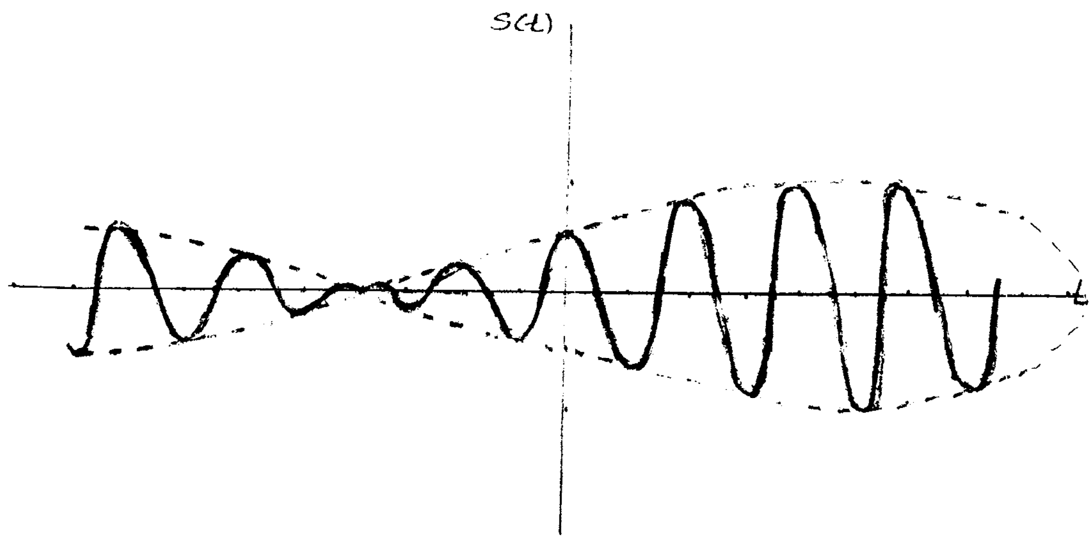
7.4

(b) For 100% modulation, $k_a=2$. Hence

$$s(t) = A_c\left(1 + \frac{2t}{1+t^2}\right)\cos(2\pi f_c t)$$

$$= A_c \frac{(1+t)^2}{1+t^2}\cos(2\pi f_c t)$$

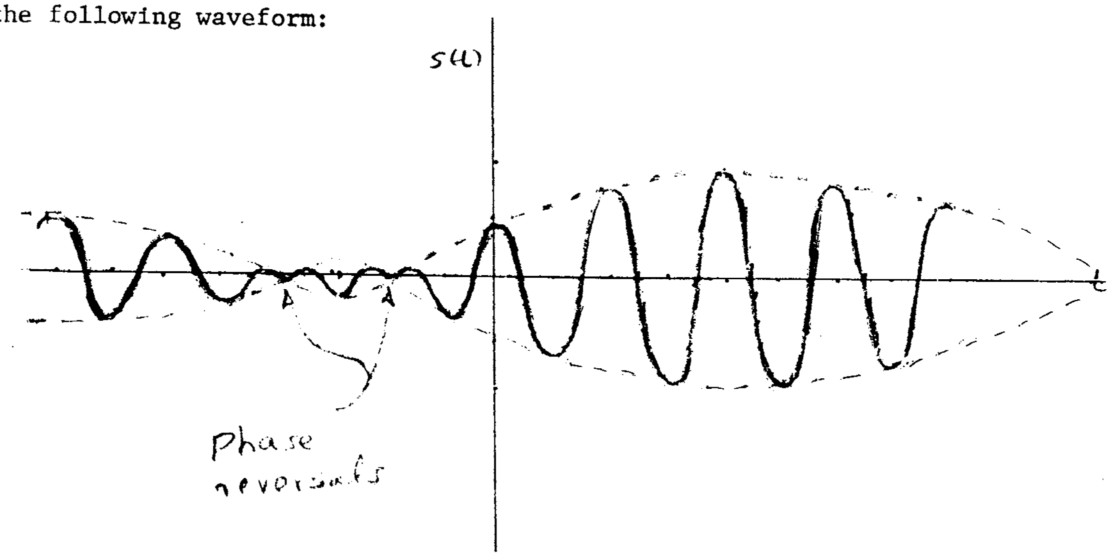
which has the following waveform:



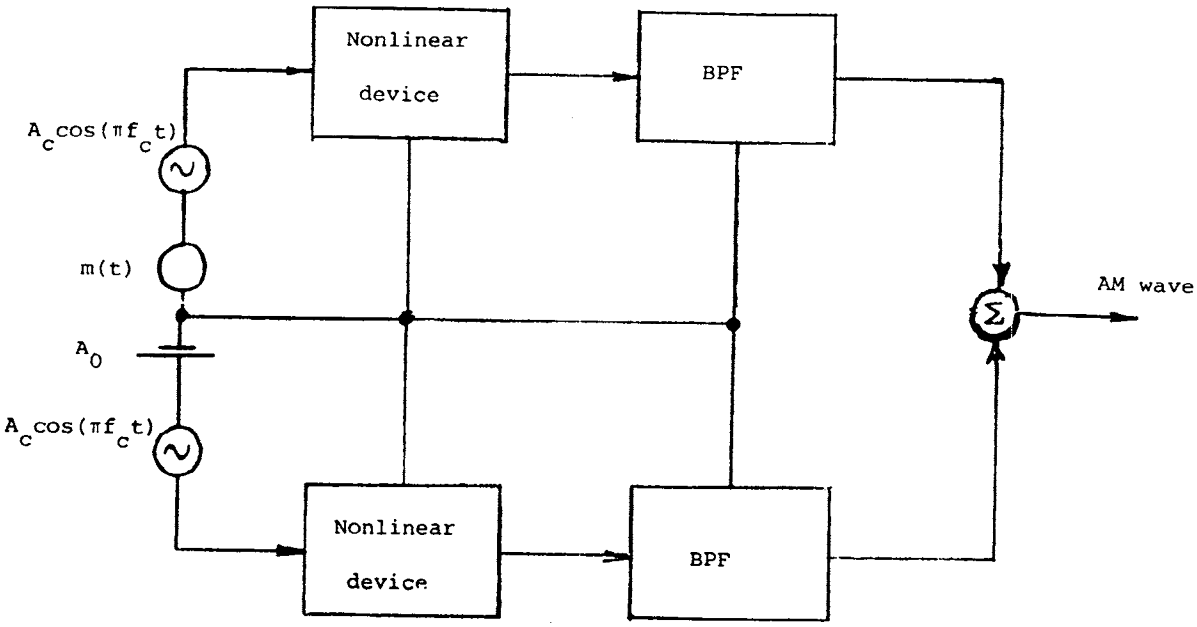
(c) For 125% modulation, $k_a=2.5$. Hence,

$$s(t) = A_c\left(1 + \frac{2.5t}{1+t^2}\right)\cos(2\pi f_c t)$$

which has the following waveform:



(2) To generate an AM wave with carrier frewuency f_c we require a sinusoidal component of frequency f_c to be added to the DSBSC generated in the manner described above. To achieve this requirement, we may use the following configuration involving a pair of the nonlinear devices and a pair of identical band-pass filters.



The resulting AM wave is therefore $\frac{3}{2}\, a_3\, A_c^2[A_0+m(t)]\cos(2\pi f_c t)$. Thus, the choice of the dc level A_0 at the input of the lower branch controls the percentage modulation of the AM wave.

√ Problem 6

By expanding the triangular envelope of the AM wave in a Fourier series, we find that the components of the envelope beyond the 11th harmonic become increasingly insignificant. This means that for negligible distortion, the envelope detector must be able to reproduce faithfully the sinusoidal components of the envelope up to the 11th harmonic. The frequency of this component is 1100 Hz. For the evenlope detector to function properly, we must therefore choose

$$R_\ell C \ll \frac{1}{W}$$

where $R_\ell = 250\ \Omega$

$W = 1100$ Hz

That is, C must be small compared to 4 μF. Thus, the value C = 0.4 μF would be acceptable.

√ Problem 7

The DSBSC modulated wave is defined by

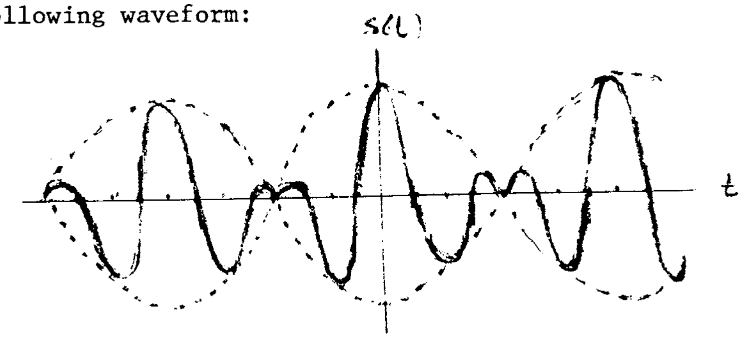
$$s(t) = m(t)\ c(t)$$

$$= A_m A_c \cos(2\pi f_m t)\cos(2\pi f_c t+\phi)$$

(a) The phase difference $\phi = 0$. Hence

$$s(t) = A_m A_c \cos(2\pi f_m t)\cos(2\pi f_c t),$$

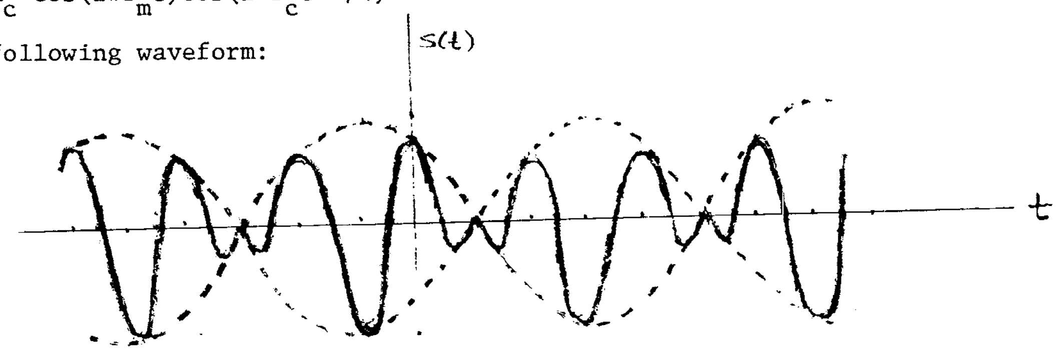
which has the following waveform:



(b) The phase difference $\phi = 45^{\circ} = \pi/4$. Hence

$$s(t) = A_m A_c \cos(2\pi f_m t)\cos(2\pi f_c t + \pi/4)$$

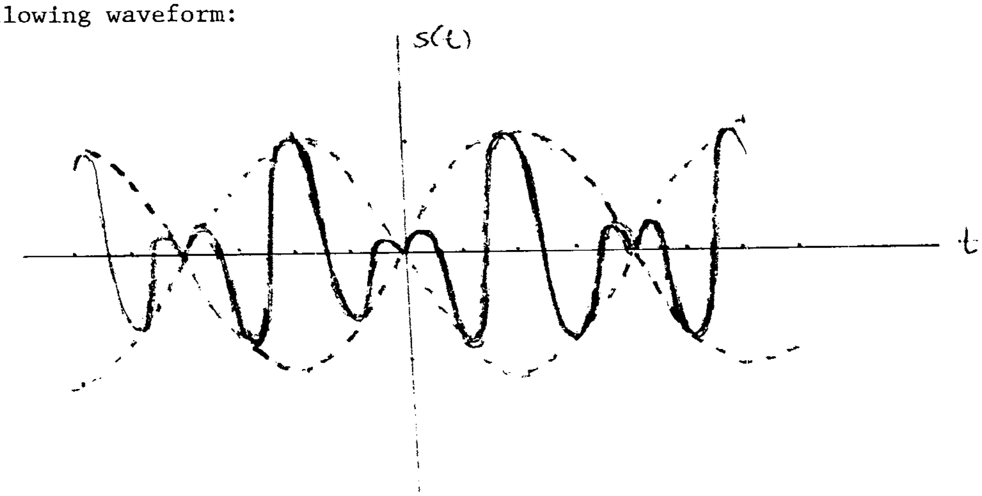
which has the following waveform:



(c) The phase difference $\phi = 90^{\circ} = \pi/2$. Hence

$$s(t) = - A_m A_c \cos(2\pi f_m t)\sin(2\pi f_c t)$$

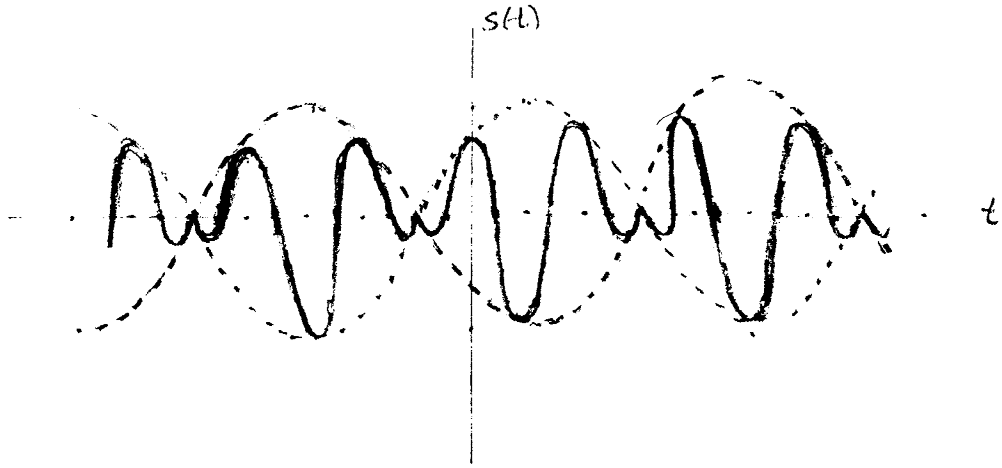
which has the following waveform:



(d) The phase difference $\phi = 135^{\circ} = 3\pi/4$. Hence

$$s(t) = A_m A_c \cos(2\pi f_m t)\cos(2\pi f_c t + 3\pi/4)$$

Correspondingly, we have the waveform:



Problem 8

Utilizing the result obtained in Problem 2, we find that, since the carrier is suppressed, the tuned filter output is equal to

$$0.125\ A_c R[\cos(2.01\pi \times 10^6 t - 1.05) + \cos(1.99\pi \times 10^6 t + 1.05)]$$

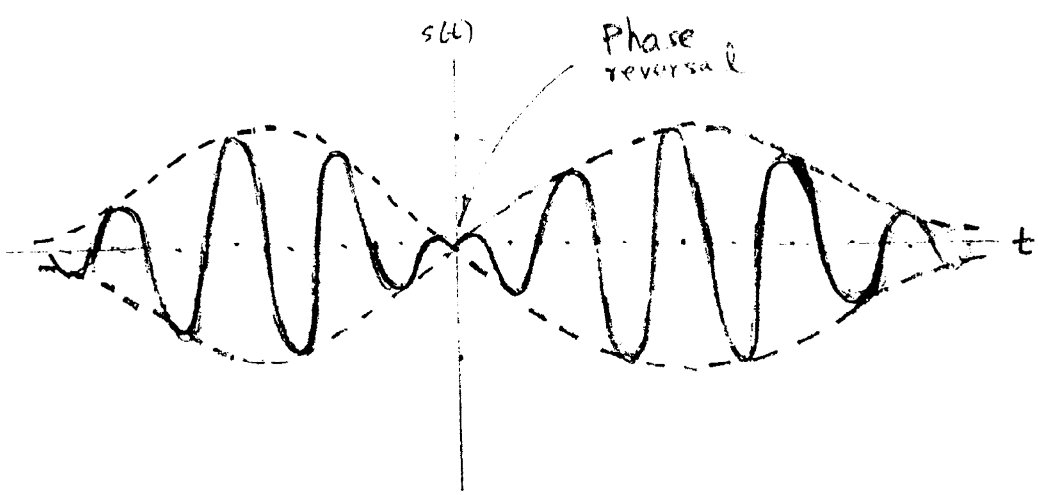
Problem 9

The DSBSC wave is defined by

$$s(t) = A_c\ m(t)\cos(2\pi f_c t)$$

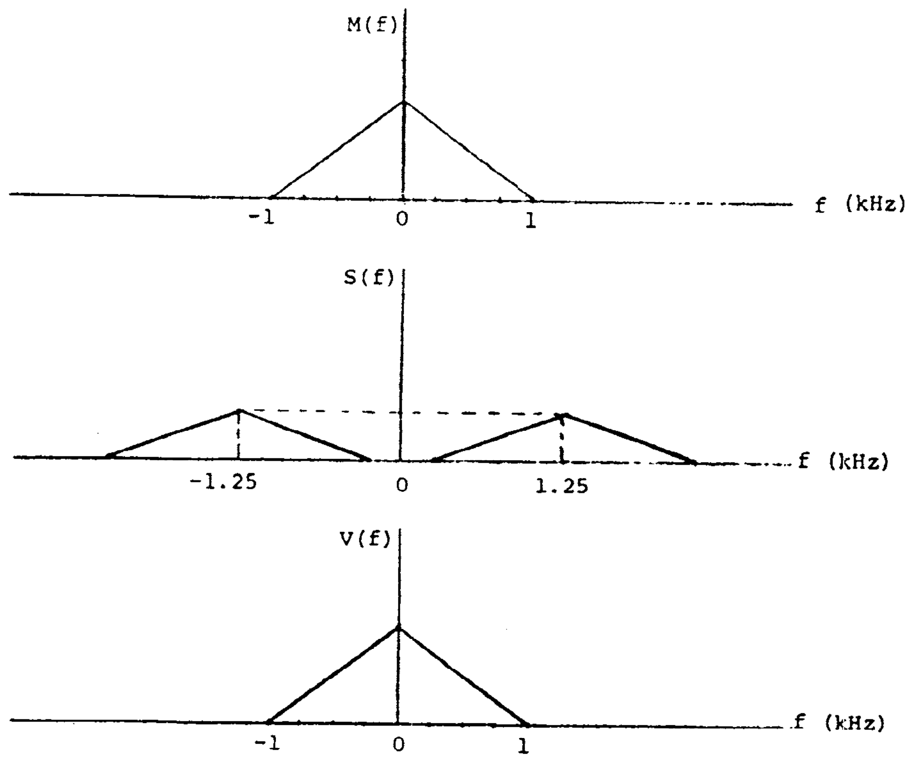
$$= A_c\left(\frac{t}{1+t^2}\right)\cos(2\pi f_c t)$$

which has the following waveform:



Problem 12

(a) For f_c = 1.25 kHz, the spectra of the message signal m(t), the product modulator output s(t), and the coherent detector output v(t) are as follows, respectively.



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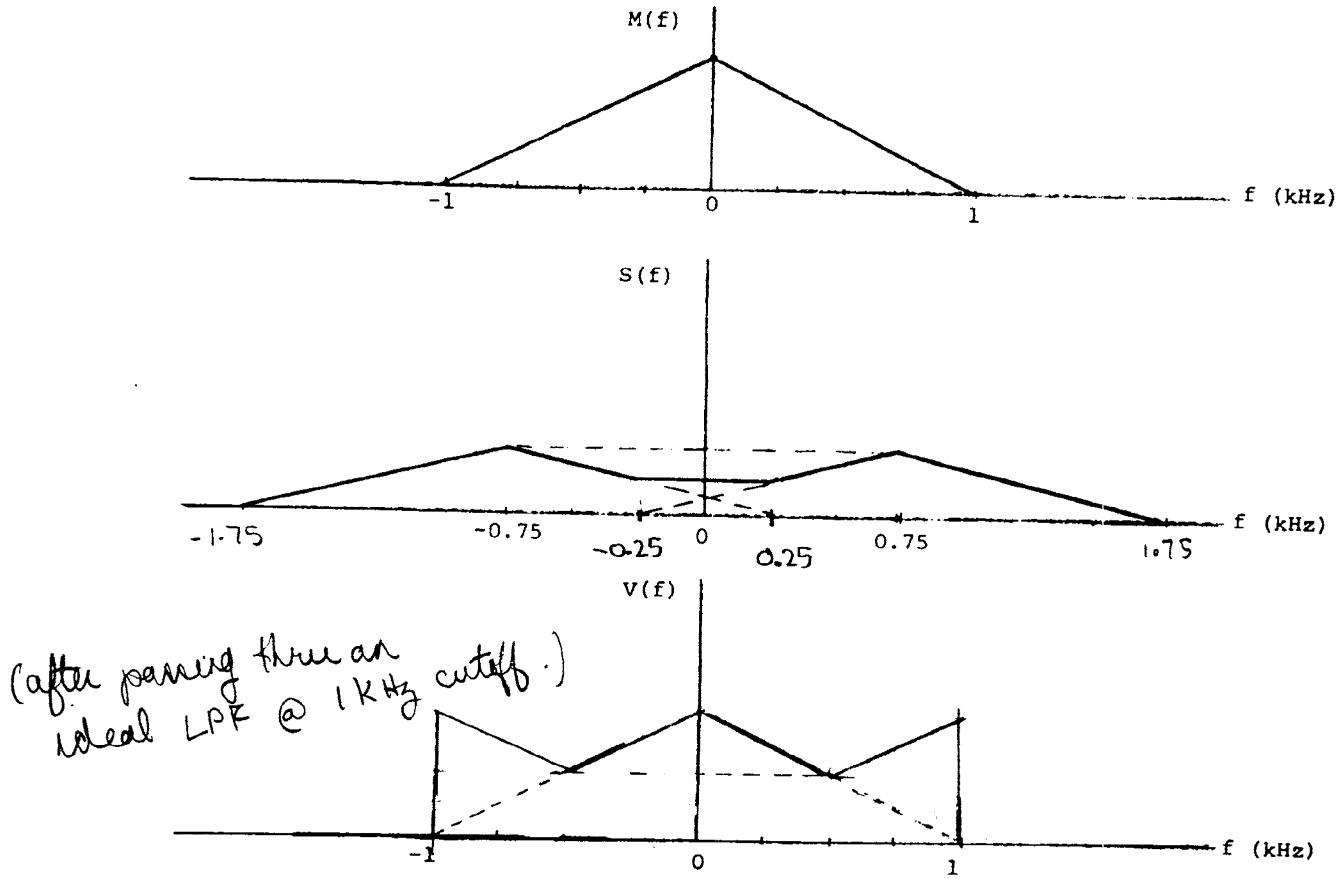
(b) For the case when $f_c = 0.75$, the respective spectra are as follows:



To avoid sideband-overlap, the carrier frequency f_c must be greater than or equal to 1 kHz, The lowest carrier frequency is therefore 1 kHz for each sideband of the modulated wave s(t) to be uniquely determined by m(t).

Problem 13

(a) Multiplying the signal by the local oscillator gives:

$$s_1(t) = A_c m(t) \cos(2\pi f_c t) \cos[2\pi(f_c+\Delta f)t]$$

$$= \frac{A_c}{2} m(t) \{\cos(2\pi\Delta ft) + \cos[2\pi(2f_c+\Delta f)t]\}$$

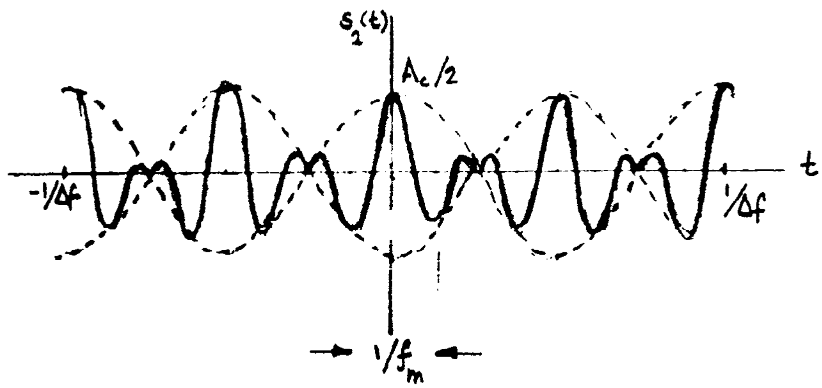
Low pass filtering of $s_1(t)$ leaves the product:

$$s_2(t) = \frac{A_c}{2} m(t) \cos(2\pi\Delta ft)$$

Thus the output signal is the message signal modulated by a sinusoid of frequency Δf.

(b) If $m(t) = \cos(2\pi f_m t)$,

then $s_2(t) = \frac{A_c}{2} \cos(2\pi f_m t) \cos(2\pi \Delta f t)$



Problem 14

$$s(t) = \cos(2\pi f_c t)m(t) + A_c \cos(2\pi f_c t+\phi)$$

$$= \cos(2\pi f_c t)m(t) + A_c \cos\phi\cos(2\pi f_c t) - A_c \sin\phi\sin(2\pi f_c t)$$

$$= [m(t)+A_c\cos\phi]\cos(2\pi f_c t) - A_c \sin\phi\sin(2\pi f_c t)$$

The envelope of s(t) is therefore

$$a(t) = \sqrt{[m(t)+A_c\cos\phi]^2+A_c^2\sin^2\phi}$$

$$= \sqrt{m^2(t)+2A_c\cos\phi m(t)+A_c^2} \qquad (1)$$

(a) When $\phi=0$, Eq. (1) reduces to

$$a(t) = A_c + m(t)$$

Except for a dc component equal to A_c, we thus find that the envelope detector output is proportional to the message signal m(t).

(b) When $\phi\neq 0$ and $|m(t)|<<A_c$, Eq. (1) may be approximated as follows

$$a(t) \simeq \sqrt{2A_c \cos\phi m(t) + A_c^2}$$

$$= A_c \sqrt{1 + \frac{2}{A_c} \cos\phi m(t)} \qquad (2)$$

Since $|\cos\phi| \leq 1$, and $|m(t)| << A_c$, we may further approximate Eq. (2) as

$$a(t) \simeq A_c[1 + \frac{\cos\phi}{A_c} m(t)]$$

$$= A_c + \cos\phi m(t)$$

Here, again, we find that except for the dc component, A_c, the envelope detector output is proportional to m(t) for a constant value of ϕ.

Problem 15

√ The multiplexed signal is

$$s(t) = A_c\, m_1(t) \cos(2\pi f_c t) + A_c\, m_2(t) \sin(2\pi f_c t)$$

Therefore,

$$S(f) = \frac{A_c}{2} [M_1(f-f_c)+M_1(f+f_c)] + \frac{A_c}{2j} [M_2(f-f_c)-M_2(f+f_c)]$$

where $M_1(f) = F[m_1(t)]$ and $M_2(f) = F[m_2(t)]$. The spectrum of the received signal is therefore

$$R(f) = H(f)S(f)$$

$$= \frac{A_c}{2} H(f) [M_1(f-f_c)+M_1(f+f_c) + \frac{1}{j} M_2(f-f_c) - \frac{1}{j} M_2(f+f_c)]$$

To recover $m_1(t)$, we multiply r(t), the inverse Fourier transform of R(f), by $\cos(2\pi f_c t)$ and then pass the resulting output through a low-pass filter, producing a signal with the following spectrum

$$F[r(t)\cos(2\pi f_c t)] = \frac{1}{2} [R(f-f_c)+R(f+f_c)]$$

$$= \frac{A_c}{4} H(f-f_c)[M_1(f-2f_c) + M_1(f) + \frac{1}{j} M_2(f-2f_c) - \frac{1}{j} M_2(f)]$$

$$+ \frac{A_c}{4} H(f+f_c)[M_1(f) + M_1(f+2f_c) + \frac{1}{j} M_2(f) - \frac{1}{j} M_2(f+2f_c)] \qquad (1)$$

The condition $H(f_c+f) = H^*(f_c-f)$ is equivalent to $H(f+f_c)=H(f-f_c)$; this follows from the fact that for a real-valued impulse response h(t), we have $H(-f)=H^*(f)$. Hence, substituting this condition in Eq. (1), we get

$$F[r(t)\cos(2\pi f_c t)] = \frac{A_c}{2} H(f-f_c)M_1(f)$$

$$+ \frac{A_c}{4} H(f-f_c)[M_1(f-2f_c) + \frac{1}{j} M_2(f-2f_c)+M_1(f+2f_c) - \frac{1}{j} M_2(f+2f_c)]$$

The low-pass filter output, therefore, has a spectrum equal to $(A_c/2)\, H(f-f_c)M_1(f)$.

Similarly, to recover $m_2(t)$, we multiply r(t) by $\sin(2\pi f_c t)$, and then pass the resulting signal through a low-pass filter. In this case, we get an output with a spectrum equal to $(A_c/2)\, H(f-f_c)M_2(f)$.

Problem 16

✓ The DSBSC modulated wave is

$$s(t) = m(t)c(t)$$

$$= 1000\cos(2\pi t)\cos(100\pi t)$$

$$= 500[\cos(102\pi t) + \cos(98\pi t)]$$

(a) When only the upper side frequency is transmitted, we have

$$s_u(t) = 500\cos(102\pi t)$$

(b) When only the lower side frequency is transmitted, we have

$$s_\ell(t) = 500\cos(98\pi t)$$

These two SSB modulated waves are both sinusoidal in waveform, differing only in frequency. Specifically, the frequency of $s_u(t)$ is 51 Hz, and that of $s_\ell(t)$ is 49 Hz.

Problem 17

The Hilbert transform of a rectangular pulse of amplitude A and duration T is given by

$$\hat{m}(t) = \frac{A}{\pi}\ln\left|\frac{\frac{t}{T}+\frac{1}{2}}{\frac{t}{T}-\frac{1}{2}}\right|$$

The pulse m(t) and its Hilbert transform $\hat{m}(t)$ are illustrated below:

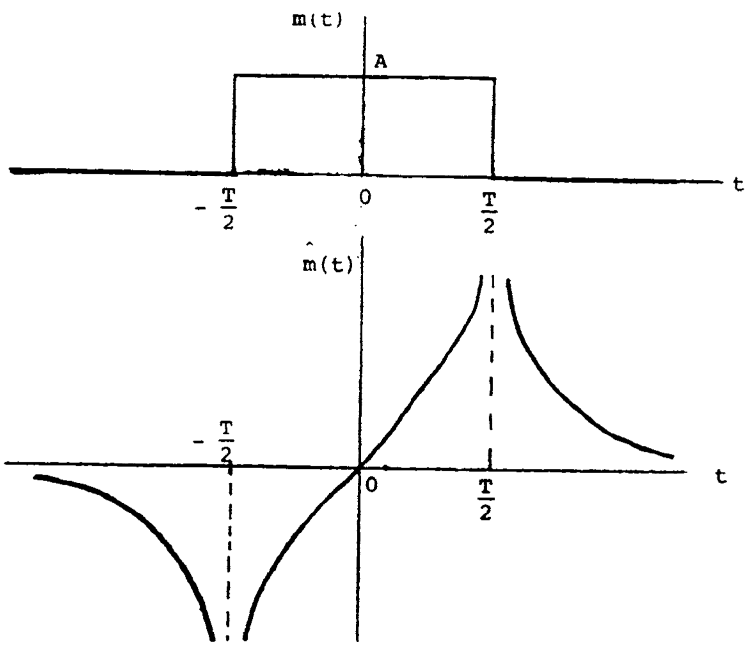


Table 1 - Summary of the Positive Frequency Bands Occupied by the Sidebands of the Modulated Waves in Fig. 7.18b

Modulated Wave	Frequency Band Occupied by the Lower Sideband	Frequency Band Occupied by the Upper Sideband
$v_1(t)$	96.6-99.7 kHz	100.3-103.4 kHz
$s_1(t)$		100.3-103.4 kHz
$v_2(t)$	9.8966-9.8997 MHz	10.1003-10.1034 MHz
$s_2(t)$		10.1003-10.1034 MHz

Problem 19

(a) The SSB wave $s_u(t)$ is defined is defined by

$$s_u(t) = \frac{A_c}{2}[m(t)\cos(2\pi f_c t) - \hat{m}(t)\sin(2\pi f_c t)]$$

and its Hilbert transform by

$$\hat{s}_u(t) = \frac{A_c}{2}[m(t)\sin(2\pi f_c t) + \hat{m}(t)\cos(2\pi f_c t)]$$

We may therefore write

$$s_u(t)\cos(2\pi f_c t) = \frac{A_c}{2}[m(t)\cos^2(2\pi f_c t) - \hat{m}(t)\sin(2\pi f_c t)\cos(2\pi f_c t)]$$

$$\hat{s}_u(t)\sin(2\pi f_c t) = \frac{A_c}{2}[m(t)\sin^2(2\pi f_c t + \hat{m}(t)\sin(2\pi f_c t)\cos(2\pi f_c t)]$$

Adding these two expressions, and solving for m(t), we get

$$m(t) = \frac{2}{A_c}[s_u(t)\cos(2\pi f_c t) + \hat{s}_u(t)\sin(2\pi f_c t)] \quad (1)$$

Next we note that

$$s_u(t)\sin(2\pi f_c t) = \frac{A_c}{2}[m(t)\cos(2\pi f_c t)\sin(2\pi f_c t) - \hat{m}(t)\sin^2(2\pi f_c t)]$$

$$\hat{s}_u(t)\cos(2\pi f_c t) = \frac{A_c}{2}[m(t)\sin(2\pi f_c t)\cos(2\pi f_c t) + \hat{m}(t)\cos^2(2\pi f_c t)]$$

Subtracting the first of these expressions from the second, and then solving for m(t), we get

$$\hat{m}(t) = \frac{2}{A_c}[\hat{s}_u(t)\cos(2\pi f_c t) - s_u(t)\sin(2\pi f_c t)]$$

(b) The SSB wave $s_\ell(t)$ is defined by

$$s_\ell(t) = \frac{A_c}{2}[m(t)\cos(2\pi f_c t) + \hat{m}(t)\sin(2\pi f_c t)],$$

and its Hilbert transform is defined by

$$\hat{s}_\ell(t) = \frac{A_c}{2}\,[m(t)\,\sin(2\pi f_c t) - \hat{m}(t)\,\cos(2\pi f_c t)]$$

Therefore,

$$s_\ell(t)\,\cos(2\pi f_c t) = \frac{A_c}{2}\,[m(t)\,\cos^2(2\pi f_c t) + \hat{m}(t)\,\sin(2\pi f_c t)\,\cos(2\pi f_c t)]$$

$$\hat{s}_\ell(t)\,\sin(2\pi f_c t) = \frac{A_c}{2}\,[m(t)\,\sin^2(2\pi f_c t) - \hat{m}(t)\,\cos(2\pi f_c t)\,\sin(2\pi f_c t)]$$

Adding these two expressions and then solving for $m(t)$, we get

$$m(t) = \frac{2}{A_c}\,[s_\ell(t)\,\cos(2\pi f_c t) + \hat{s}_\ell(t)\,\sin(2\pi f_c t)] \qquad (2)$$

Next, we note that

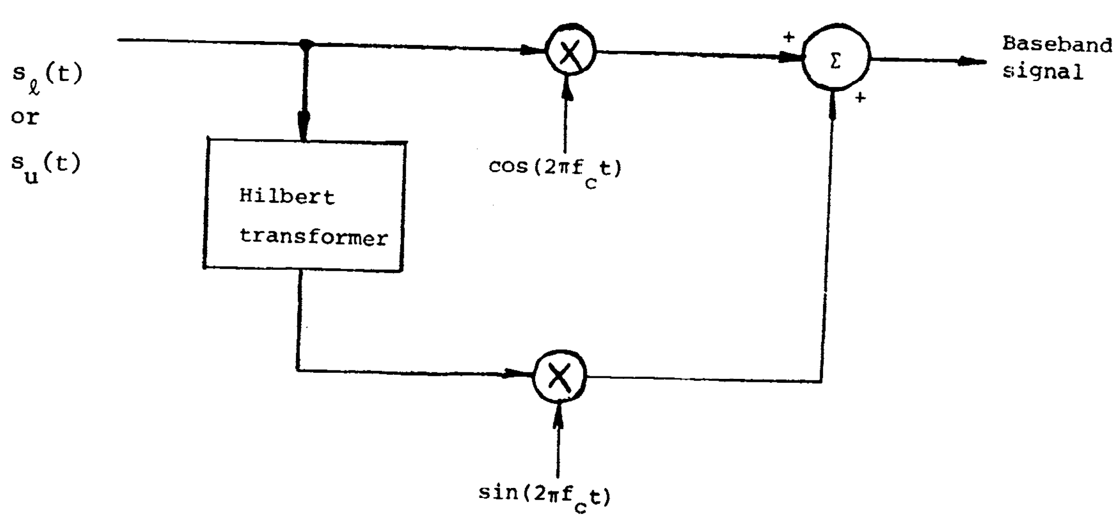
$$s_\ell(t)\,\sin(2\pi f_c t) = \frac{A_c}{2}\,[m(t)\,\cos(2\pi f_c t)\,\sin(2\pi f_c t) + \hat{m}(t)\,\sin^2(2\pi f_c t)]$$

$$\hat{s}_\ell(t)\,\cos(2\pi f_c t) = \frac{A_c}{2}\,[m(t)\,\sin(2\pi f_c t) - \hat{m}(t)\,\cos^2(2\pi f_c t)]$$

Subtracting the second expression from the first one and then solving for $m(t)$, we get

$$\hat{m}(t) = \frac{2}{A_c}\,[s_\ell(t)\,\sin(2\pi f_c t) - \hat{s}_\ell(t)\,\cos(2\pi f_c t)]$$

(c) From Eqs. (1) and (2), we see that the baseband signal $m(t)$ may be recovered from $s_u(t)$ or $s_\ell(t)$ by using the following arrangement:



Problem 20

(a) The frequency error $\Delta f=20$ Hz. Since this frequency error is positive and the incoming SSB wave contains the upper sideband, the frequency components of the demodulated signal are shifted inward by Δf, compared with the baseband signal. The demodulated signal therefore consists of three frequency components: 80, 180, and 380 Hz.

(b) When the lower sideband is transmitted, the frequency components of the demodulated signal are shifted outward by Δf, compared with the baseband signal. The demodulated signal, therefore, consists of three frequency components: 120, 220, and 420 Hz.

Problem 21

The VSB modulated wave is

$$\begin{aligned} s(t) &= a\,A_mA_c\,\cos[2\pi(f_c+f_m)t] + A_mA_c(1-a)\cos[2\pi(f_c-f_m)t] \\ &= a\,A_mA_c[\cos(2\pi f_m t)\cos(2\pi f_c t) - \sin(2\pi f_m t)\sin(2\pi f_c t)] \\ &\quad + A_mA_c(1-a)[\cos(2\pi f_m t)\cos(2\pi f_c t) + \sin(2\pi f_m t)\sin(2\pi f_c t)] \\ &= A_mA_c\,\cos(2\pi f_m t)\cos(2\pi f_c t) - A_mA_c(2a-1)\sin(2\pi f_m t)\sin(2\pi f_c t) \end{aligned} \tag{1}$$

(a) The in-phase component of the VSB modulated wave is

$$S_I(t) = A_mA_c\,\cos(2\pi f_m t),$$

and its quadrature component is

$$S_Q(t) = A_mA_c(2a-1)\sin(2\pi f_m t).$$

(b) If $a=1/2$, the $s(t)$ of Eq.(1) reduces to the DSBSC modulated wave

$$s(t) = A_mA_c\,\cos(2\pi f_m t)\cos(2\pi f_c t)$$

(c) For $a=0$, the $s(t)$ of Eq.(1) reduces to an SSB wave containing only the lower side-frequency, as shown by

$$s_\ell(t) = +\,A_mA_c\,\cos[2\pi(f_c-f_m)t]$$

For $a=1$, it reduces to an SSB wave containing only the upper side-frequency, as shown by

$$s_u(t) = A_mA_c\,\cos[2\pi(f_c+f_m)t]$$

(d) Adding the carrier to the VSB modulated wave, the envelope detector input is

$$\begin{aligned} s'(t) &= s(t) + A_c\,\cos(2\pi f_c t) \\ &= A_c[1 + A_m\,\cos(2\pi f_m t)]\cos(2\pi f_c t) - A_mA_c(2a-1)\sin(2\pi f_m t)\sin(2\pi f_c t) \end{aligned}$$

The envelope detector output is therefore

$$a(t) = \sqrt{A_c^2[1 + A_m\,\cos(2\pi f_m t)]^2 + A_m^2A_c^2(2a-1)^2\sin^2(2\pi f_m t)}$$