ECE 712 Assignment 1 2009

Due: Monday Nov. 23, 2009

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This is a graded assignment and as such you are expected to do this work on your own without consultation with any other person.

1. Consider the arbitrary square full rank matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and an augmented version of it, $\boldsymbol{A}_1 = [\boldsymbol{A} \;\; \boldsymbol{I}]$, where $\boldsymbol{I}$ is the $n \times n$ identity matrix. Perform Gauss transforms on $\boldsymbol{A}_1$ so that the result is $\boldsymbol{A}_2 = [\boldsymbol{U} \;\; \boldsymbol{B}_1]$, where $\boldsymbol{U}$ is upper triangular. What is $\boldsymbol{B}_1$?

   Now suggest a modified Gauss transform $\boldsymbol{N}_i, \; i = n, \ldots, 2$ so that

   $$\boldsymbol{A}_3 = \left(\prod_{i=n}^{2} \boldsymbol{N}_i \right) \boldsymbol{A}_2 = [\boldsymbol{D} \;\; \boldsymbol{B}_2] \tag{1}$$

   where $\boldsymbol{D}$ is diagonal. Explain your procedure in detail.

   Now produce a matrix $\boldsymbol{A}_4$ using simple row operations on $\boldsymbol{A}_3$, to convert $\boldsymbol{D}$ to the identity; i.e.,

   $$\boldsymbol{A}_4 = \boldsymbol{D}^{-1}[\boldsymbol{D} \;\; \boldsymbol{B}_2] = [\boldsymbol{I} \;\; \boldsymbol{B}_3]. \tag{2}$$

   What is $\boldsymbol{B}_3$? Explain your reasoning.

2. Prove that the diagonal elements of a symmetric positive–definite matrix are positive.

3. Consider the matrix $\boldsymbol{A}$ as in Q1, so that $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$. It is possible to compute $\boldsymbol{A}^{-1}$ column–wise, by solving a sequence of linear equations. Explain this procedure, and then recommend a computationally efficient method of solving the systems of equations for this purpose. *Hint: you are solving repeated systems of equations with the same matrix $\boldsymbol{A}$ but different right–hand sides.*

4. Provide an analytical closed–form solution to the problem

   $$\boldsymbol{x}_{\text{opt}} = \arg\min ||\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}||_2^2 + \gamma||\boldsymbol{x}||_2^2 \tag{3}$$

   where $\gamma \in \mathbb{R}$, $\boldsymbol{b} \in \mathbb{R}^m$ and $\boldsymbol{A} \in \mathbb{R}^{m \times n}, m \geq n$, not necessarily full rank. Give some intuition on this formulation for the case when $\boldsymbol{A}$ is poorly conditioned.

   Load the file assig1Q4 2009 from the course website to give variables $\boldsymbol{A}$ and $\boldsymbol{b}$ pertinent to this question. Examine the effect of γ on the $||\boldsymbol{x}_{\text{opt}}||_2$ obtained by your solution and comment. By experimentation in Matlab, propose a value of γ for which $||\boldsymbol{x}_{\text{opt}}||_2 \leq 1$.

5. Show that the rank of a projector matrix is equal to its trace.