Lecture 11 Toeplitz Systems

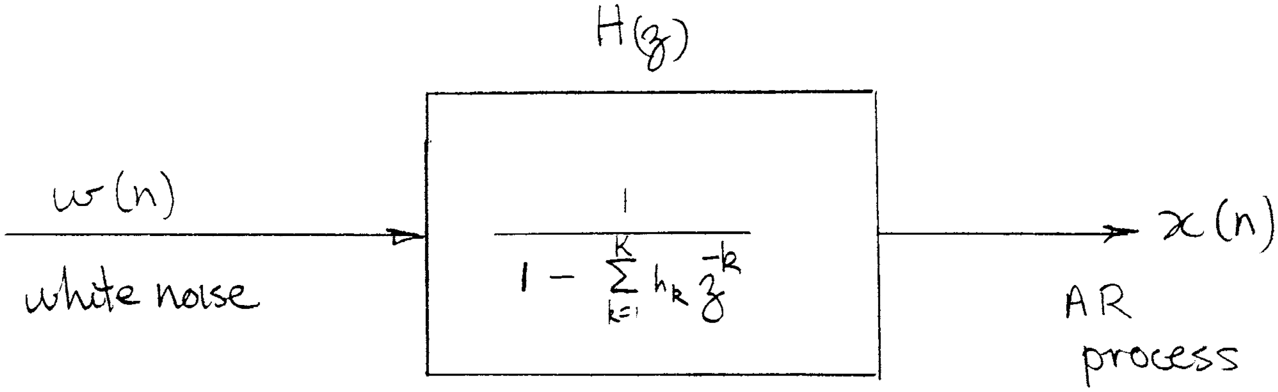
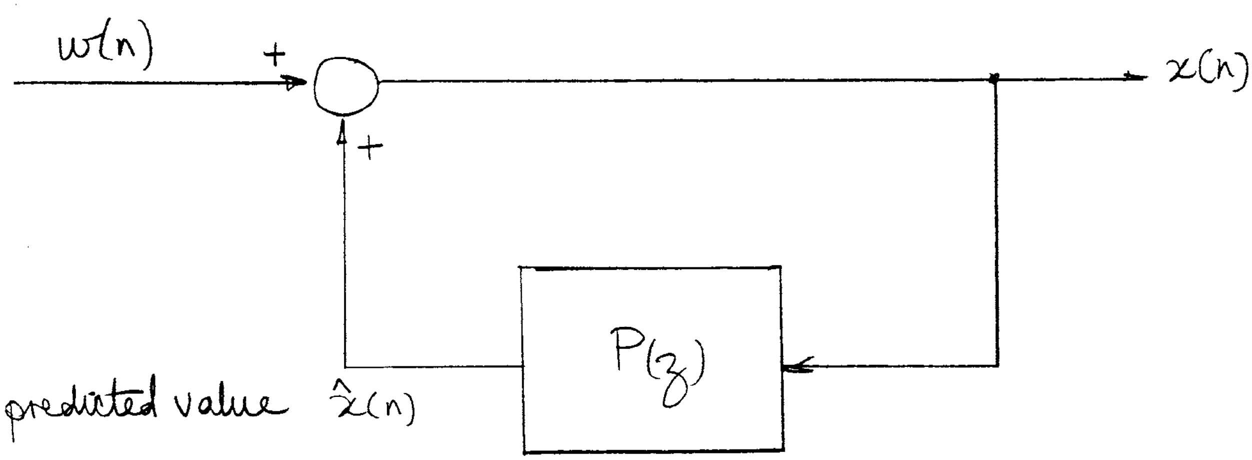


Fig. 1. The generation of an AR process.



$$P(z) = \sum_{k=1}^{K} h_k z^{-k}$$

Fig. 2. Alternate interp. of Fig. 1, emphasizing the prediction process.

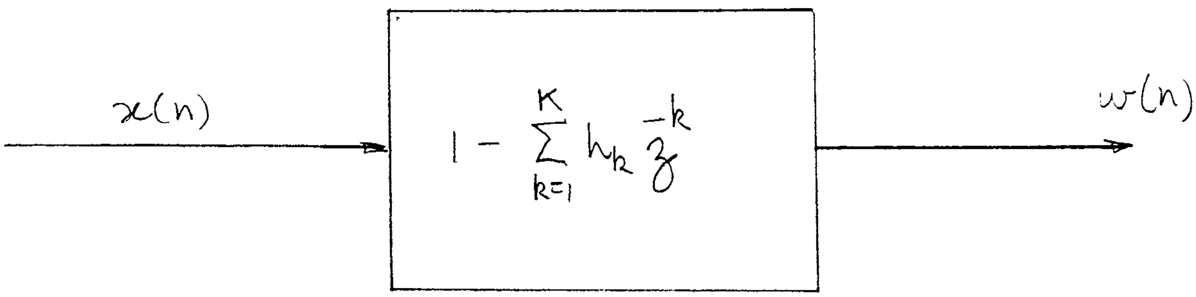


Fig. 3a. Forward prediction error filter Basic configuration. This operation is the inverse to that of Fig. 1.

$n = 1, \ldots, N.$

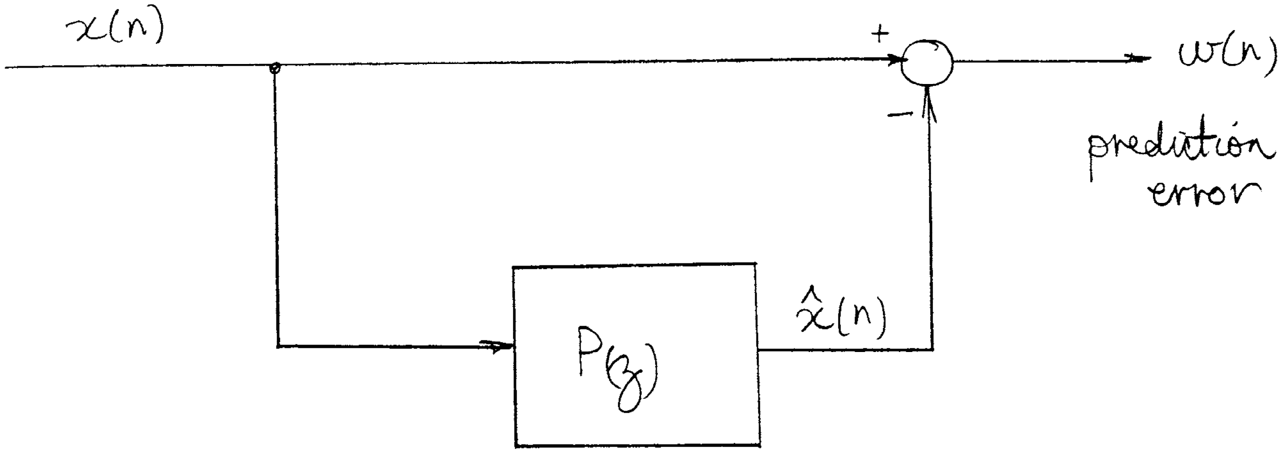


Fig. 3b. Forward PEF. This construction emphasizes that $w(n)$ is the error between the predicted value $\hat{x}(n)$ and $x(n)$. $x(n)$ is an AR process "matched" to $P(z)$.

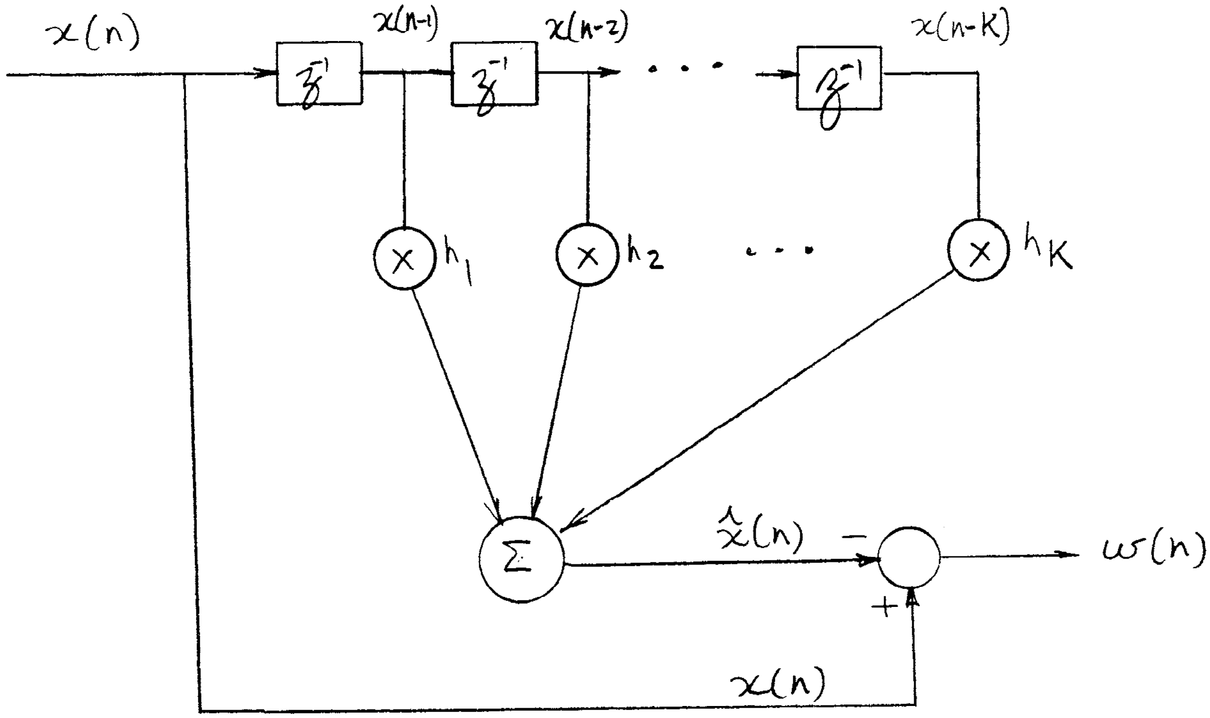


Fig. 3c. Tapped delay-line configuration of the PEF.

The input is in time-reversed order

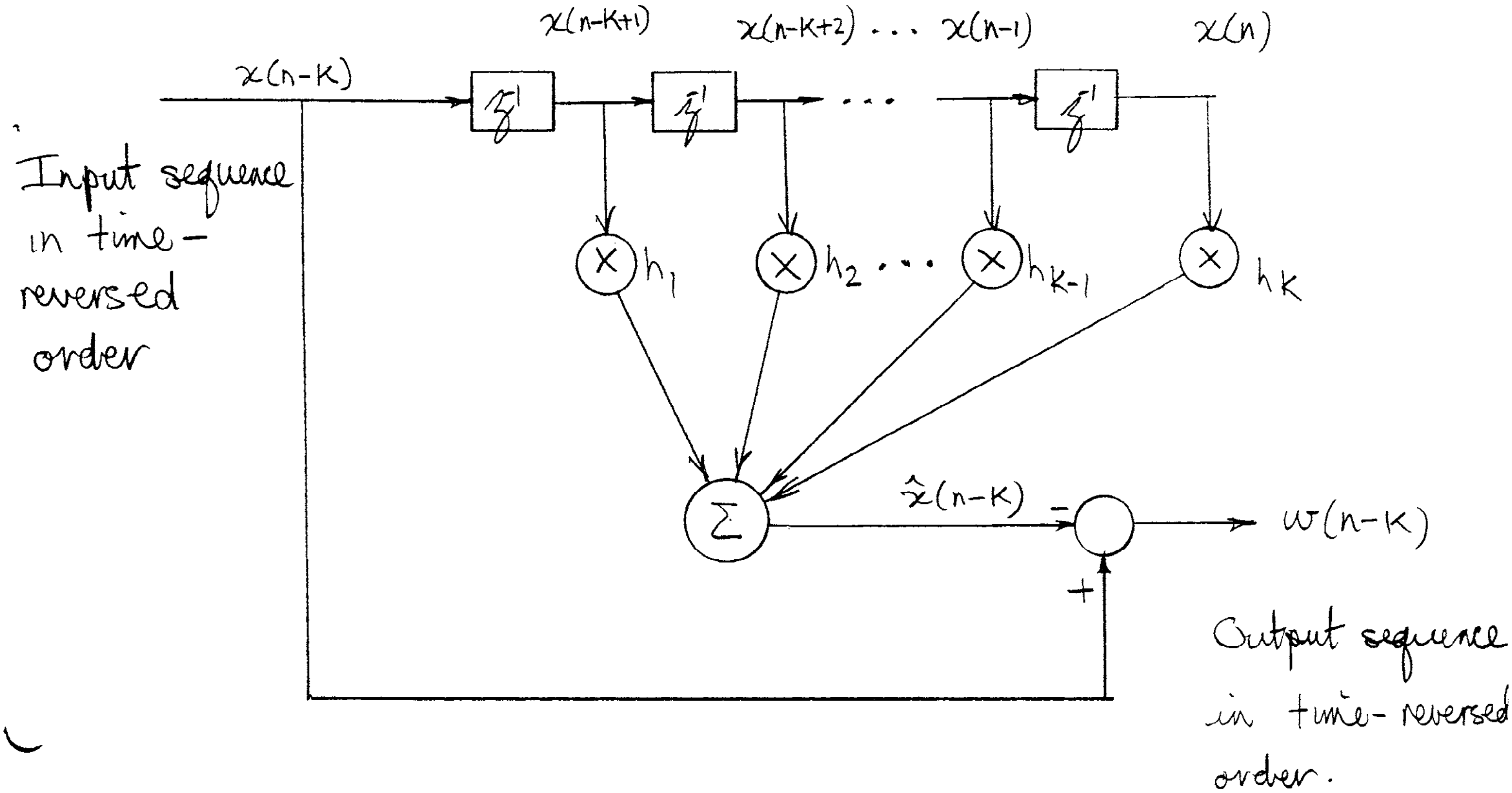


Fig. 4a. Backward PEF. The input sequence $x(n)$ is time-reversed, and hence prediction is into the past. The coefficients h_k, $k=1,\dots K$ are identical to those of the forward case.

the input is in time-ascending order

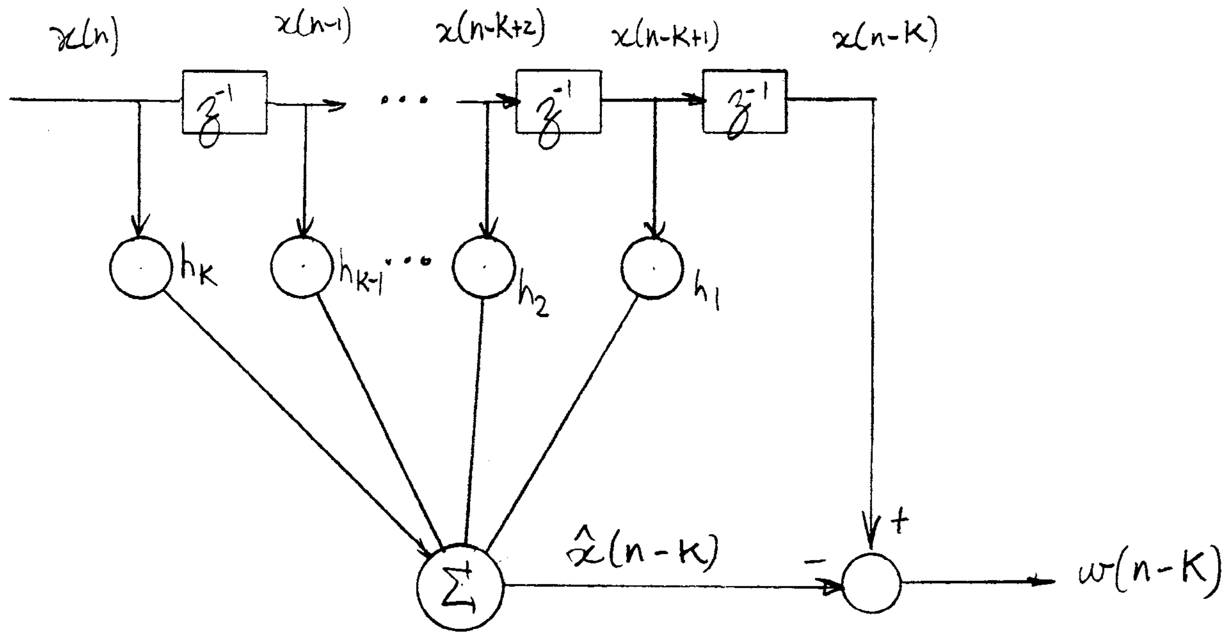


Fig. 4b. Backward PEF. Both sequences $x(n)$ & $h(k)$ are time-reversed with respect to those of Fig. 4a. (consequently, so is $\{w(n)\}$.) Prediction is still in the backward direction (into the past) but the input sequence is now in the usual time-ascending order. Other than the ordering of the input sequences, the PEF's of Figs. 4a & 4b. are identical.

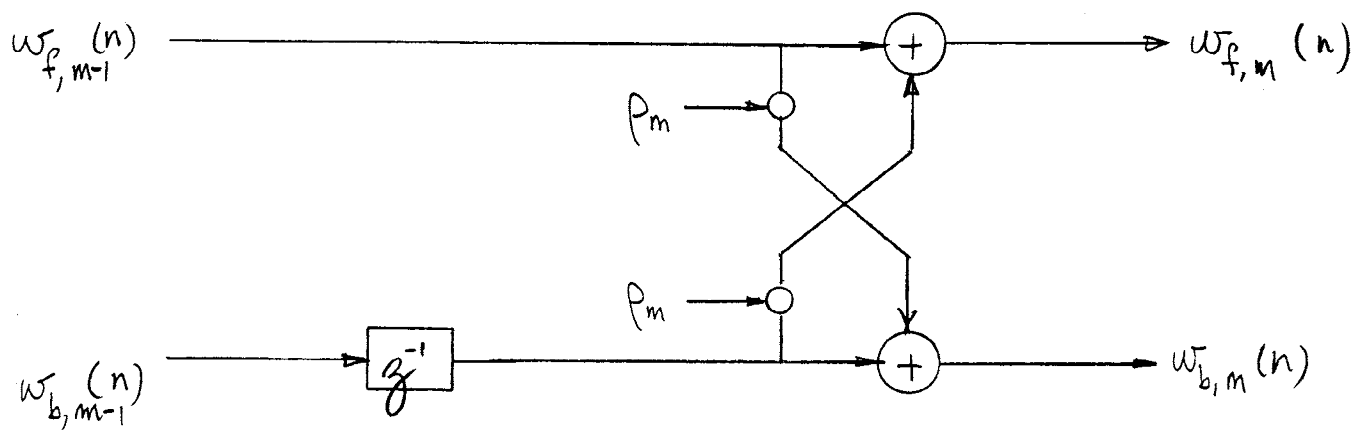


Fig. 5. One stage of the lattice-filter equivalent of the tapped-delay line PEF structure corresponding jointly to eqs. 40 & 43.

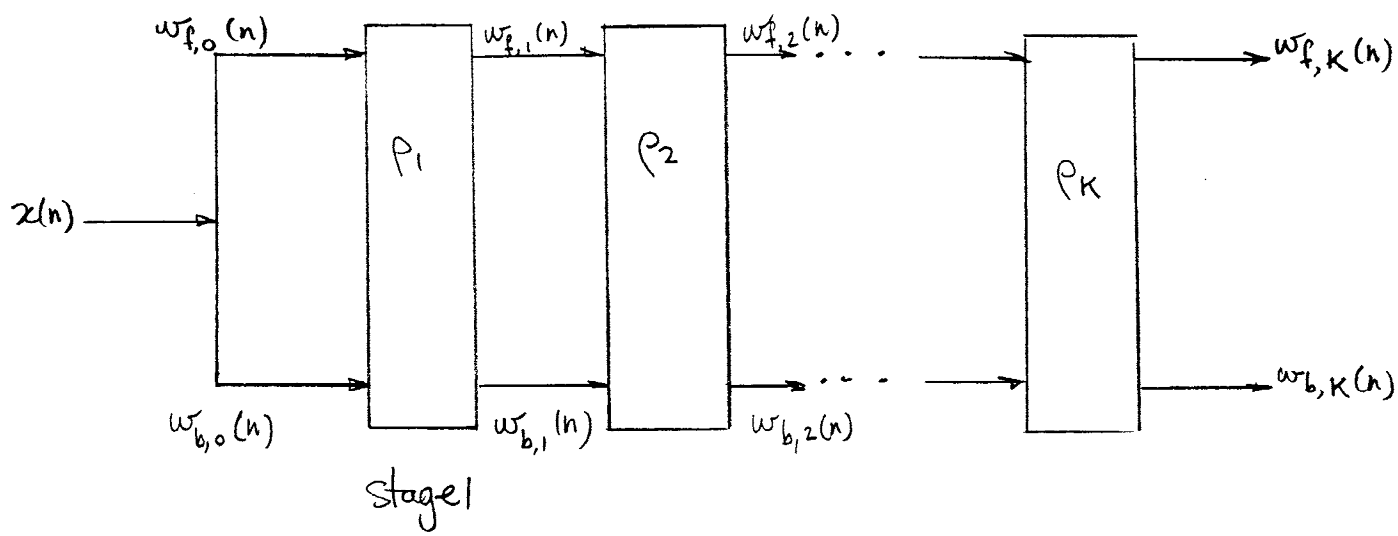


Fig. 6. Cascade of stages of Fig. 5 to realize the complete forward/backward PEF structure. The content of each box is the same as Fig. 5.

$$\begin{pmatrix} r_0 & r_{-1} & \cdots & r_{-(m-1)} \\ r_1 & r_0 & & \vdots \\ \vdots & & \ddots & \vdots \\ r_{m-1} & & & r_0 \end{pmatrix} \begin{pmatrix} 1 \\ -h_1 \\ -h_2 \\ \vdots \\ -h_{m-1} \end{pmatrix} = \begin{pmatrix} \sigma_{m-1}^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

fwd. prediction-error eqs. of order m-1. (eq 14 in notes).

$$\begin{pmatrix} r_0 & r_{-1} & \cdots & r_{-(m-1)} \\ r_{-1} & r_0 & & \\ \vdots & & \ddots & \\ r_{m-1} & \cdots & & r_0 \end{pmatrix} \begin{pmatrix} -h_{m-1} \\ \vdots \\ -h_2 \\ -h_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sigma_{m-1}^2 \end{pmatrix}$$

bwd. prediction-error eqs. of order m-1. (eq. 17 in notes).

$$\begin{pmatrix} r_0 & r_{-1} & \cdots & r_{-m} \\ r_1 & r_0 & & \vdots \\ \vdots & & \ddots & \vdots \\ r_m & \cdots & & r_0 \end{pmatrix} \begin{pmatrix} 1 \\ -h_1 \\ -h_2 \\ \vdots \\ -h_m \end{pmatrix} = \begin{pmatrix} \sigma_m^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad (1)$$

fwd. prediction-error eqs. of order m.

How to go from (m-1)th order system to an mth-order system by taking advantage of the Toeplitz structure of $\underline{R}$?

We hypothesize by induction that the mth-order coeffs. can be written in terms of the (m-1)th-order coeffs. as:

$$\begin{pmatrix} 1 \\ -h_1^{(m)} \\ \vdots \\ -h_m^{(m)} \end{pmatrix} = \begin{pmatrix} 1 \\ -h_1^{(m-1)} \\ \vdots \\ -h_{m-1}^{(m-1)} \\ 0 \end{pmatrix} + \rho_m \begin{pmatrix} 0 \\ -h_{m-1}^{(m-1)} \\ \vdots \\ -h_1^{(m-1)} \\ 1 \end{pmatrix} \qquad (2)$$

where ρ_m is to be determined.

Substituting (2) into (1) for an mth-order system we have:

$$\begin{pmatrix} r_0 & r_{-1} & \cdots & r_{-m} \\ r_1 & r_0 & & \vdots \\ \vdots & & \ddots & \\ r_m & \cdots & & r_0 \end{pmatrix} \left\{ \overset{\text{fwd.}}{\begin{pmatrix} 1 \\ -h_1^{(m-1)} \\ \vdots \\ -h_{m-1}^{(m-1)} \\ 0 \end{pmatrix}} + \rho_m \overset{\text{bwd.}}{\begin{pmatrix} 0 \\ -h_{m-1}^{(m-1)} \\ \vdots \\ -h_1^{(m-1)} \\ 1 \end{pmatrix}} \right\}$$

$$= \left\{ \overset{\text{fwd.}}{\begin{pmatrix} \sigma_m^2 \\ 0 \\ \vdots \\ 0 \\ \Delta^{(m-1)} \end{pmatrix}} + \rho_m \overset{\text{bwd.}}{\begin{pmatrix} \Delta^{(m-1)} \\ 0 \\ \vdots \\ 0 \\ \sigma_m^2 \end{pmatrix}} \right\} \qquad (3)$$

Every eq. of (3) except for the last line of the fwd. part occurs in the corresponding $(m-1)$ th order forward system.

Likewise every eq. except for the first of the bwd. part occurs in the corresponding $(m-1)$ th order backward system.

These are the same equation; let them equate to the qt'y $\Delta^{(m-1)}$.

$$\Delta^{(m-1)} = - \sum_{i=m}^{1} r_i \, h^{(m-1)}(m-i)$$

Let us equate corresponding components of (1) & (3):

$$h_i^{(m)} = h_i^{(m-1)} + \rho_m h_{m-i}^{(m-1)} \qquad i = 2, \cdots m-1$$

$$h_m^{(m)} = -\rho_m .$$

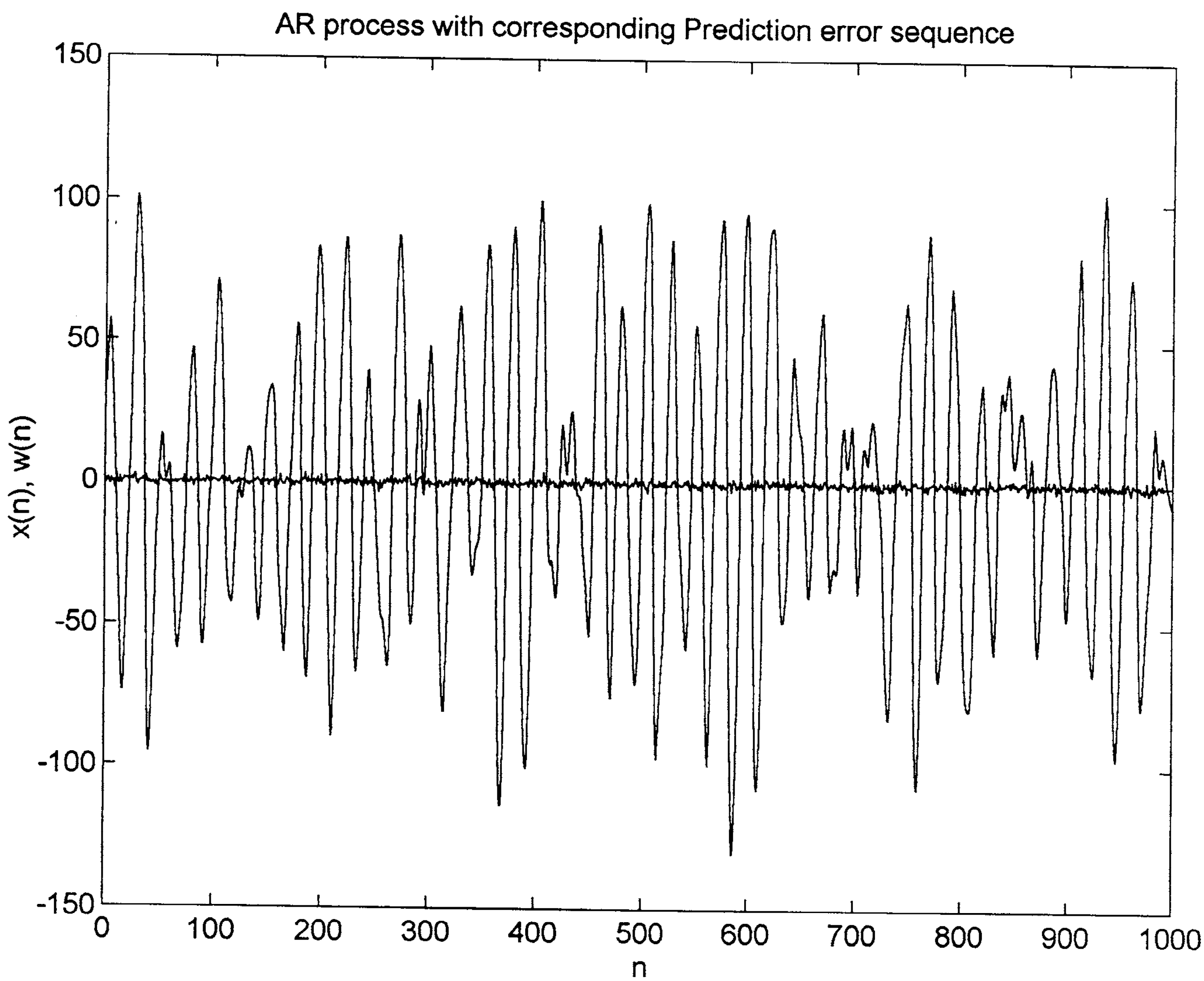
$$h_1^{(m)} = 1.$$

$$\sigma^2_{(m-1)} + \rho_m \Delta^{(m-1)} = \sigma^2_m$$

$$\Delta^{(m-1)} + \rho_m \sigma^2_m = 0$$

$$\rho_m = \frac{\Delta^{(m-1)}}{\sigma^2_{(m-1)}}$$

$$\sigma^2_m = \sigma^2_{m-1} \left(1 - \rho_m^2\right).$$


AR process with corresponding Prediction error sequence
150
100
50
0
-50
-100
-150
x(n), w(n)
0
100
200
300
400
500
600
700
800
900
1000
n