# ECE 712 Homework on Eigenvalues/Eigenvectors

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**1.** Consider a skew–symmetric matrix (i.e., one for which $\boldsymbol{A}^T = -\boldsymbol{A}$). Prove its eigenvalues are pure imaginary, and the eigenvectors are mutually orthogonal.

**2.** Consider the system of equations $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A}$ is full rank, square and symmetric. Give a closed–form expression for the solution $\boldsymbol{x}$ in terms of the eigendecomposition of $\boldsymbol{A}$.

**3.** Repeat Q2, but where $\boldsymbol{A}$ is rank deficient, and $\boldsymbol{b} \in R(\boldsymbol{A})$. Verify your closed–form expression using matlab, on a $5 \times 5$ square symmetric matrix of rank 3. (For this, you will have to construct such a matrix using matlab.)

**4.** With reference to Q3, what happens if $\boldsymbol{b}$ not $\in R(\boldsymbol{A})$? Suggest a reasonable closed–form solution for this case.

**5.** Consider both systems of equations in Q2 and Q3. Suggest a change of basis in each case for $\boldsymbol{x}$ and $\boldsymbol{y}$ so that the system becomes diagonal.

**6.** . Consider two square symmetric matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. Give a sufficient condition on these matrices so that $\boldsymbol{AB} = \boldsymbol{BA}$.

**7.** (a.) What are the eigenvectors of a diagonal matrix where none of the diagonal elements are equal?

(b.) as in (a), but where some of the diagonal elements are equal?

(c.) as in (a), but where all of the diagonal elements are equal.

**8.** Prove that the eigendecomposition of a square symmetric matrix $\boldsymbol{A} = \boldsymbol{V\Lambda V}^T$ can be written as $\boldsymbol{A} = \sum_{i=1}^{n} \lambda_i \boldsymbol{v}_i \boldsymbol{v}_i^T$.

**9.** Consider the so–called power method for calculating the largest eigenvalue/vector pair of a square symmetric matrix $\boldsymbol{A}$. The algorithm can be described as follows:

initialize: set $\boldsymbol{x}_0$ to an arbitrary value.

for $i = 1, 2, 3, \ldots,$

$\boldsymbol{z} = \boldsymbol{A}\boldsymbol{x}_{i-1}$

$$\boldsymbol{x}_i = \frac{\boldsymbol{z}}{||\boldsymbol{z}||_2}$$

As $i \to \infty$, $\boldsymbol{x}$ converges to the maximum eigenvector.

**a.** Prove it converges. *Hint:* express $\boldsymbol{x}_0$ using the eigenvectors of $\boldsymbol{A}$ as a basis.

**b.** Modify the algorithm to find the smallest eigenvalue/vector.

**c.** Explain how to accelerate convergence in part (b).

**d.** Explain what happens when the largest eigenvalue is not distinct.

**10.** We are given a matrix $\boldsymbol{A}$ whose eigendecomposition is $\boldsymbol{A} = \boldsymbol{V\Lambda V^T}$. Find the eigenvalues and eigenvectors of the matrix $\boldsymbol{C} = \boldsymbol{BAB}^{-1}$ in terms of those of $\boldsymbol{A}$, where $\boldsymbol{B}$ is any invertible matrix. $\boldsymbol{C}$ is referred to as a *similarity transform* of $\boldsymbol{A}$.

**11.** We are given two random process $x$ and $y$. We form corresponding matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$, both of which are in $\Re^{m\times n}$, $m \geq 2n$, from $x$ and $y$ respectively, in the manner described in the Ch.2 notes. We then form the matrix $\boldsymbol{Z} = \begin{bmatrix}\boldsymbol{X} & \boldsymbol{Y}\end{bmatrix}$. The covariance matrix $\boldsymbol{R}_{zz}$ of $\boldsymbol{Z}$ has the form

$$\boldsymbol{R}_{zz} = \begin{bmatrix} \boldsymbol{R}_{xx} & \boldsymbol{R}_{xy} \\ \boldsymbol{R}_{yx} & \boldsymbol{R}_{yy} \end{bmatrix}. \tag{1}$$

Suggest transformations $\boldsymbol{T}_x \in \Re^{n\times n}$ and $\boldsymbol{T}_y \in \Re^{n\times n}$ on $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively, such that $\tilde{\boldsymbol{X}} = \boldsymbol{X}\boldsymbol{T}_x$ and $\tilde{\boldsymbol{Y}} = \boldsymbol{Y}\boldsymbol{T}_y$, so that the corresponding $\boldsymbol{R}_{\tilde{z}\tilde{z}}$ has the form

$$\boldsymbol{R}_{\tilde{z}\tilde{z}} = \begin{bmatrix} \boldsymbol{I}_{n\times n} & \boldsymbol{\Sigma} \\ \boldsymbol{\Sigma} & \boldsymbol{I}_{n\times n} \end{bmatrix}, \tag{2}$$

where $\tilde{\boldsymbol{Z}} = \begin{bmatrix}\tilde{\boldsymbol{X}} & \tilde{\boldsymbol{Y}}\end{bmatrix}$, and $\boldsymbol{\Sigma} \in \Re^{n\times n}$ is diagonal. The diagonal elements of $\boldsymbol{\Sigma}$ are called the *canonical correlation coefficients* of the processes $x$ and $y$. Show that these coefficients have values between zero and one.

**12.** Consider a non–white noise process $x[k]$ of duration $m$ with a known covariance matrix $\boldsymbol{\Sigma}$. The sequence $f[k]$ of duration $n \ll m$ operates on $x[k]$ to give an output sequence $y[k]$ according to

$$y[k] = \sum_i f[i]x[k-i].$$

**a.** Show that this operation (convolution) can be expressed as a matrix–vector multiplication.

**b.** Find $f[k]$ so that $\left|\left|y[k]\right|\right|_2$ is minimized, subject to $||\boldsymbol{f}||_2 = 1$.

**13.** Show that the diagonal elements of a positive definite (not necessarily symmetric) matrix must be positive.

**14.** Derive an analytical expression for the eigendecomposition of a square symmetric rank–one matrix.