ECE 712 Take–home final exam 2006

Due: Wednesday Dec. 6, 2006 at 4:00pm.

James P. Reilly

Please hand in completed exam to Cheryl Gies in ITB A112. If you have a valid reason, the due date may be extended. If you wish to apply for an extended deadline, please send me an email giving all the relevant details. In this case you will be asked to give your word that by delaying the due date, you will not receive any undue advantage over other students.

Cheryl will be keeping a list of the total number of hours each student spends on this exam. Please keep track of this figure and let Cheryl know. Her list will not include names, in case it is perceived these figures will bias the marking. I would like to know these total times, so that I can assess the difficulty of this exam for future students.

This exam is based on the honour system. As such, please sign the following statement:

This exam is entirely my own work. I may consult any written material or text, but I have NOT received nor offered any direct or indirect help from any other person.

Name: Date:

Signature:

Given that individuals may be sitting closely together in the same lab, doing this exam at the same time, it may be difficult to avoid discussion of this exam. I therefore urge you avoid this situation and work e.g., in the library. It is your responsibility to work in an environment where you can honour the above oath.

1. Find the solution to the following optimization problem:

$$\min_{\boldsymbol{Q}^{-1}} \left[-\sum_{i=1}^{N} \left(\boldsymbol{A}\boldsymbol{x_i} - \boldsymbol{b}_i\right)^T \boldsymbol{Q}^{-1} \left(\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{b}_i\right) + \ln \det \boldsymbol{Q}^{-1} \right] \tag{1}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $m \geq n$, $\boldsymbol{x}_i \in \mathbb{R}^n, \boldsymbol{b} \in \mathbb{R}^m$ and $\boldsymbol{Q} \in \mathbb{R}^{m \times m}$. $\boldsymbol{A}$ and $\boldsymbol{Q}$ are full rank.

2. A QR decomposition process on $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $m \geq n$ has proceeded i stages, where $i < n - 1$. An appropriate pivoting strategy is used in the decomposition. Give an expression for $||\boldsymbol{a}_k||_2$ in terms of the $\boldsymbol{R}$ that exists at the ith stage, for $k \le i$ and $k > i$. For the latter case, express $\boldsymbol{a}_k$ as two components: $\boldsymbol{a}_{s,k} \in \text{span}[\boldsymbol{a}_1 \ldots, \boldsymbol{a}_i]$ and $\boldsymbol{a}_{\perp,k} \perp \text{span}[\boldsymbol{a}_1 \ldots, \boldsymbol{a}_i]$. Use this to justify the column pivoting strategy discussed in class.

3. At a certain time t, we have available m row vectors $\boldsymbol{a}_i^T \in \mathbb{R}^n$ and their corresponding desired values b_i, for $i = 1, \ldots, m$, to form the matrix $\boldsymbol{A}_t \in \mathbb{R}^{m \times n}$ and $\boldsymbol{b}_t \in \mathbb{R}^m$. The QR decomposition $\boldsymbol{Q}_t^T \boldsymbol{A}_t = \boldsymbol{R}_t$ is available at time t to aid in the computation of the LS problem $\min_{\boldsymbol{x}} ||\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}||$. At time $t+1$ a new $(m+1)$th row $\boldsymbol{a}_{m+1}^T$ of $\boldsymbol{A}$ and a new $(m+1)$th element of $\boldsymbol{b}$ become available. Explain in detail how to update $\boldsymbol{Q}_t$ and $\boldsymbol{R}_t$ to get $\boldsymbol{Q}_{t+1}$ and $\boldsymbol{R}_{t+1}$. *Hint:* $\boldsymbol{A}_{t+1}$ can be decomposed as

$$\boldsymbol{A}_{t+1} = \begin{bmatrix} \boldsymbol{Q}_t & \boldsymbol{z} \\ \boldsymbol{z}^T & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{R}_t \\ \boldsymbol{a}_{m+1}^T \end{bmatrix}$$

where $\boldsymbol{z}$ is an $m \times 1$ vector of zeros. Give an estimate of the order of the FLOP count for this process.

4. On the course website you will find a file Q4exam.mat, which contains a matrix $\boldsymbol{A}$ and a corresponding column $\boldsymbol{b}$ which define an LS problem. Solve this problem in matlab using the *complete orthogonal decomposition* (COD). Compare with the pseudo–inverse solution.

5. (From Demmel) Let $\boldsymbol{x} \in \mathbb{R}^n$ and let $\boldsymbol{H}$ be a Houselholder matrix such that $\boldsymbol{H}\boldsymbol{x} = \pm||\boldsymbol{x}||_2\boldsymbol{e}_1$. Let $\boldsymbol{G}_{1,2}, \ldots, \boldsymbol{G}_{n-1,n}$ be Givens rotations, and let $\boldsymbol{Q} = \boldsymbol{G}_{1,2}\boldsymbol{G}_{2,3}\ldots, \boldsymbol{G}_{n-1,n}$. Suppose $\boldsymbol{Q}\boldsymbol{x} = \pm||\boldsymbol{x}||_2\boldsymbol{e}_1$. Must $\boldsymbol{Q} = \boldsymbol{H}$? (You need to give a proof or a counterexample).