ECE 712 Homework on Matrix Fundamentals

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1. Using Matlab, construct a 6×5 matrix $\boldsymbol{A}$ of rank 3. Use this matrix for questions 2 - 4 below.

2. Using the null($\cdot$) command in Matlab, find an orthonormal basis for the nullspace of $\boldsymbol{A}$. Call the corresponding subspace $\mathcal{N}$. Show, using Matlab, that a vector $\boldsymbol{v} \in \mathcal{N}$ is orthogonal to any row $\boldsymbol{a}_j^T, j = 1, \ldots, 6$.

3. Find an orthonormal basis for $R(\boldsymbol{A})_\perp$. Call the corresponding subspace $\mathcal{Q}$. Show, using Matlab, that a vector $\boldsymbol{v} \in \mathcal{Q}$ is orthogonal to any column $\boldsymbol{a}_i, i = 1, \ldots, 5$.

4. What condition(s) must exist on $\boldsymbol{b}$ so that the system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ has an exact solution? Demonstrate an example using Matlab.

5. Assume $\mathbf{u}, \mathbf{v} \in \mathbb{R}^\mathbf{n}$ and $\mathbf{A} = \mathbf{I} + \mathbf{uv^T}$. Show that if $\mathbf{A}$ is nonsingular, then $\mathbf{A^{-1}} = \mathbf{I} + \alpha\mathbf{uv^T}$, for some scalar α. Find the corresponding α.

6. Show that $\max_i |\lambda_i| \leq \|\mathbf{A}\|$ where λ_i are the eigenvalues of $\mathbf{A}$ and $\|.\|$ is any matrix norm.

7. Let $\boldsymbol{A} \in \Re^{m \times n}$ for $m > n$. The QR decomposition of $\boldsymbol{A}$ can be written as

$$\boldsymbol{A} = \boldsymbol{QR} \tag{1}$$

where $\boldsymbol{Q}$ is an $m \times m$ orthonormal matrix and

$$\boldsymbol{R} = \begin{bmatrix} \tilde{\boldsymbol{R}} \\ \mathbf{0} \end{bmatrix} \tag{2}$$

where $\tilde{\boldsymbol{R}}$ is an $n \times n$ upper triangular matrix. Let $\boldsymbol{A}$ be rank p and assume that $\tilde{\boldsymbol{R}}$ is arranged so that $|r_{11}| \geq |r_{22}| \geq \ldots \geq |r_{nn}|$. Identify an orthonormal basis for $R(\boldsymbol{A})$ and $R(\boldsymbol{A})_\perp$. Identify the shape of $\tilde{\boldsymbol{R}}$ when $p \leq n$. Using the QR decomposition, find an orthonormal basis for $N(\boldsymbol{A})$.

8. Define the *Kronecker product* of two matrices as

$$\boldsymbol{A} \otimes \boldsymbol{B} = \begin{bmatrix} a_{11}\boldsymbol{B}, & a_{12}\boldsymbol{B}, & \ldots & a_{1n}\boldsymbol{B} \\ a_{21}\boldsymbol{B}, & \ldots & \ldots & a_{2n}\boldsymbol{B} \\ \vdots & & & \vdots \\ a_{m1}\boldsymbol{B} & & \ldots & a_{mn}\boldsymbol{B} \end{bmatrix} \tag{3}$$

and the $vec(\cdot)$ operator as

$$vec\boldsymbol{A} = [\boldsymbol{a}_1^T, \boldsymbol{a}_2^T, \ldots, \boldsymbol{a}_n^T]^T \tag{4}$$

where the $\boldsymbol{a}_i$ are the *columns* of $\boldsymbol{A}$. If $\boldsymbol{Q} = \boldsymbol{Q}_1 \otimes \boldsymbol{Q}_2$, then prove that

$$\big(vec(\boldsymbol{X})\big)^T \boldsymbol{Q}\ vec(\boldsymbol{X}) = \operatorname{tr}\Big[\boldsymbol{X}^T \boldsymbol{Q}_2 \boldsymbol{X} \boldsymbol{Q}_1^T\Big], \tag{5}$$

where $\boldsymbol{X}, \boldsymbol{Q}_1, \boldsymbol{Q_2}$ are dimensionally compatible so that (5) holds.

Hint: for any matrix $\boldsymbol{A} \in \Re^{m \times m}$ and any $\boldsymbol{x} \in \Re^m$, then

$$\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} = \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} x_i x_j. \tag{6}$$