ECE 712 Assignment on the QR decomposition

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1. Go to the website www.ece.mcmaster.ca/faculty/reilly/EE2CI4. The userid is "ee2ci4", and the password is, appropriately enough, "mac is hot". (Note these entries must be lower case and include the spaces). At the bottom of the page, you will find the file Q5.mat. This file contains matrices $\boldsymbol{A}$ and $\boldsymbol{G}$, and the vector $\boldsymbol{b}$. The quantity $\boldsymbol{b}$ is given as $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{v}$, where $\boldsymbol{v}$ is noise with zero mean and covariance $\boldsymbol{\Sigma} = \boldsymbol{G}^T\boldsymbol{G}$. Using this file, find the optimal solution $\boldsymbol{x}$ which minimizes $||\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}||^2_{\Sigma^{-1}}$.

2. From the QR decomposition, a matrix $\boldsymbol{Z}$ can be written as

$$\boldsymbol{Z} = \boldsymbol{Q}\boldsymbol{R} \tag{1}$$

where $\boldsymbol{Q} \in \Re^{m\times m}$ is orthonormal, and $\boldsymbol{R} \in \Re^{m\times(n+k)}$ is upper triangular. The QR decomposition of $\boldsymbol{Z}$ can be partitioned as

$$\boldsymbol{Z} = \begin{matrix} m \end{matrix} \underset{\begin{matrix} n & m-n \end{matrix}}{\begin{bmatrix} \boldsymbol{Q}_1 & \boldsymbol{Q}_2 \end{bmatrix}} \underset{\begin{matrix} n & k \end{matrix}}{\begin{bmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \\ \boldsymbol{0} & \boldsymbol{R}_{22} \end{bmatrix}} \begin{matrix} m \end{matrix} \tag{2}$$

The matrix $\boldsymbol{Z}$ can be partitioned as

$$\boldsymbol{Z} = \begin{matrix} m \end{matrix} \underset{\begin{matrix} n & k \end{matrix}}{\begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \end{bmatrix}} \tag{3}$$

Express the projection of $\boldsymbol{A}_2$ onto $\boldsymbol{A}_1$ using only the terms in eq.(2).

3. We discussed in class the QR method for computing the eigenvalues of a square symmetric matrix $\boldsymbol{A}$. The method proceeds as follows:

initialize: $\boldsymbol{A} = \boldsymbol{A}_0 = \boldsymbol{Q}_0\boldsymbol{R}_0$
for $i = 1, 2, \ldots,$

$\boldsymbol{A}_i = \boldsymbol{R}_{i-1}\boldsymbol{Q}_{i-1}$

$\boldsymbol{A}_i = \boldsymbol{Q}_i\boldsymbol{R}_i$

endfor

The method is greatly accelerated by first converting $\boldsymbol{A}$ to tridiagonal form by an orthonormal similarity transform; i.e., we find an orthonormal $\boldsymbol{Q}$ so that $\tilde{\boldsymbol{A}} = \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T$ is nonzero only on the main and first upper and lower diagonals. The QR procedure above then operates on the matrix $\tilde{\boldsymbol{A}}$ instead of $\boldsymbol{A}$.

Write a matlab program which performs the tridiagonalization process on an arbitrary square symmetric matrix. Explain why you chose your particular procedure. What QR decomposition technique would you use for the QR method above, when the matrix is tridiagonal? Explain.

4. At a certain time t, we have available m row vectors $\boldsymbol{a}_i^T \in \mathbb{R}^n$ and their corresponding desired values b_i, for $i = 1, \ldots, m$, to form the matrix $\boldsymbol{A}_t \in \mathbb{R}^{m\times n}$ and $\boldsymbol{b}_t \in \mathbb{R}^m$. The QR decomposition $\boldsymbol{Q}_t^T\boldsymbol{A}_t = \boldsymbol{R}_t$ is available at time t to aid in the computation of the LS problem $\min_{\boldsymbol{x}} ||\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}||$. At time $t+1$ a new $(m+1)$th row $\boldsymbol{a}_{m+1}^T$ of $\boldsymbol{A}$ and a new $(m+1)$th element of $\boldsymbol{b}$ become available. Explain in detail how to update $\boldsymbol{Q}_t$ and $\boldsymbol{R}_t$ to get $\boldsymbol{Q}_{t+1}$ and $\boldsymbol{R}_{t+1}$. *Hint:* $\boldsymbol{A}_{t+1}$ can be decomposed as

$$\boldsymbol{A}_{t+1} = \begin{bmatrix} \boldsymbol{Q}_t & \boldsymbol{z} \\ \boldsymbol{z}^T & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{R}_t \\ \boldsymbol{a}_{m+1}^T \end{bmatrix}$$

where $\boldsymbol{z}$ is an $m \times 1$ vector of zeros. Give an estimate of the order of the FLOP count for this estimate.