A Stochastic EM Algorithm for Nonlinear State Estimation with Model Uncertainties

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ABSTRACT

In most solutions to state estimation problems like, for example target tracking, it is generally assumed that the state evolution and measurement models are known a priori. The model parameters include process and measurement matrices or functions as well as the corresponding noise statistics. However, there are situations where the model parameters are not known a priori or are known only partially (i.e., with some uncertainty). Moreover, there are situations that the measurement is biased. In these scenarios, standard estimation algorithms like Kalman filter and extended Kalman Filter (EKF), which assume perfect knowledge of the model parameters, are not accurate anymore. The problem with uncertain model parameters is considered as a special case of maximum likelihood estimation with incomplete-data, for which a standard solution called the expectation-maximization (EM) algorithm exists. In this paper a new extension to the EM algorithm is proposed to solve the more general problem of joint state estimation and model parameter identification for nonlinear systems with possibly non-Gaussian noise. In the expectation (E) step, it is shown that the best variational distribution over the state variables is the conditional *posterior distribution* of states given all the available measurements and inputs. Therefore, a particular type of particle filter is used to estimate and update the posterior distribution. In the maximization (M) step the nonlinear measurement process parameters are approximated using a nonlinear regression method for adjusting the parameters of a mixture of Gaussians (MofG). The proposed algorithm is used to solve a nonlinear bearing-only tracking problem similar to the one reported recently[12] with uncertain measurement process. It is shown that the algorithm is capable of accurately tracking the state vector while identifying the unknown measurement dynamics. Simulation results show the advantages of the new technique over standard algorithms like the EKF that loose the track very rapidly in uncertain model scenario.

Keywords: Nonlinear estimation, system identification, expectation-maximization algorithm, MCMC, Particle Filter, Nonlinear Regression

1. INTRODUCTION

In most solutions to state estimation problems, both linear and nonlinear, it is generally assumed that the state evolution process and measurement process models are known a priori. For instance, in target tracking using nonlinear state space model in white Gaussian noise; the extended Kalman filter (EKF) assumes that the process and measurement matrices as well as corresponding noise statistics are known.[8] However, there are situations in which the model parameters are not known a priori or are known only with some degree of uncertainty. Unknown noise covariance matrices and uncertain state and measurement models are some examples. In these circumstances, standard estimation algorithms, which are based on perfect knowledge of the model parameters, are not accurate anymore.

The problem with uncertain model parameters is considered as a special case of maximum likelihood estimation with incomplete-data, for which a standard solution called the expectation-maximization (EM) algorithm exists.[1]

In[2] a general framework for solving the general estimation/identification of linear Gaussian models are presented. Additionally, in[9] a recursive EM-type method is proposed to solve the generally *nonlinear* estimation/identification problem in the presence of model uncertainties. In this approach, using an extended Kalman

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smoother the state sequence is updated in the expectation phase, and then in the maximization phase, the model parameters are identified using this "suspicious" smoothed state data, with an optimization algorithm for fitting Gaussian radial-basis functions on the smoothed data. It is shown that the likelihood function increases in each iteration of the algorithm and after a number of iterations the algorithm attains the locally optimum solution. The noise is assumed to be Gaussian and furthermore because of using an extended Kalman smoother, this method can fail if the initial model parameters are not selected appropriately. Besides, the Gaussian assumption on noise statistics makes the algorithm unable to deal with non-Gaussian nose models.

In *stochastic* version of the EM algorithm, a stochastic algorithm is used to perform the necessary approximations in the E-step.[13] A major limitation of the EM algorithm is that whilst convergence to a stationary point of the likelihood function can be shown, this is not necessarily the global maximum. The motivation for the stochastic EM algorithm is to overcome this limitation. Two stochastic EM algorithm of interest are the SEM algorithm[3] and the MCEM algorithm.[4] In MCEM, an analytic calculation of the E-step is replaced by a Monte Carlo approximation. In SEM, the Stochastic Imputation Principle is applied applied to simulate the unobserved data based on the observed data and the current value of the parameters. There are other researches on different approximations of E-step. For instance, for computing the necessary expectations over conditional posterior distributions, Laplace approximation has been used.[6]

Besides approximating the E-step analytical solutions, there is an algorithm reported in which the maximization in M-step is performed by Monte Carlo methods.[5] Although in this algorithm an analytical E-step is developed and therefore there is no need to use Monte Carlo methods for approximating the E-step, but in general the computational expense for stochastic M-step as well as E-step is two-fold for this approach. The price for this complexity is paid to obtain a type of EM algorithm that achieves the global maximum of observation likelihood and has no dependency on initial point of the algorithm.[5] However, assuming a good initial point for the algorithm a sparse representation of data in modelling the uncertain dynamics in M-step can provide a faster computations, as in the case of the algorithm reported in this paper.

Still, there are other category of state-estimation with uncertain models in which robust state-space filters are considered. Robust in the sense that they attempt to limit, in certain ways, the effect of model uncertainties on the overall filter performance. Interested readers are referred to[7] and the references therein.

In this paper a new extension to the EM algorithm is proposed to solve the more general problem of joint state estimation and model parameter identification for nonlinear systems with possibly non-Gaussian noise. It is shown that in the E-step the best variational distribution over hidden states is the posterior distribution given the observations and input data. Then, similar to the MCEM algorithm,[3] a particular type of particle filter[11] algorithm is used to estimate and update the posterior distribution of the hidden states given the measurements. The implemented particle filter is in fact a sequential realization on Metropolis-Hasting Monte Carlo Markov Chain (M-H MCMC)[11] algorithm to update the approximated posterior *pdf*. Furthermore, in the maximization (M) step the state space model for measurement process is approximated by fitting a mixture of Gaussian (MofG) on the estimated data. The parameters of MofG are computed by maximum likelihood estimation. The sparse structure of the MofG is used to reduce the complexity of necessary representation of data in M-step. In m-step, the Gaussian assumption for the mixture kernels provides an analytical solution of maximization step. This can be realized when the log-likelihood of the Gaussian kernels simplifies to a quadratic form that provides a simple analytical solution to the optimization problem.

In Section (2) the motivation for using EM for solving the problem is given. In Section (3) the general framework for the EM algorithm is introduced. The detail for implementation of the E-step using a particular type of the MCMC method,[11] and of the M-step using fitting of a MofG on estimated data is provided in Sections (4) and (5), respectively. Then, in Section (6) the proposed method is applied to solve a nonlinear tracking problem similar to the one in[12] although with uncertain model parameters. Furthermore, in Section (6.2) an online version of the algorithm is developed and it is shown that the online version is a combination of a particle filter and a recursive least square (RLS) estimator. Simulation results are reported in Section (7).

2. NONLINEAR STATE ESTIMATION USING EM

Tracking a moving object can be modelled by a state-space dynamical system. The main idea is to estimated the state data using a sequence of noisy measurements. A general framework for the dynamical system is[8]:

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k, \boldsymbol{u}_k) + \boldsymbol{w}_k \tag{1}$$

$$\boldsymbol{z}_k = \boldsymbol{h}(\boldsymbol{x}_k, \boldsymbol{u}_k) + \boldsymbol{v}_k \tag{2}$$

where $\boldsymbol{w}_k$ and $\boldsymbol{v}_k$ are assumed to be i.i.d noise processes and $\boldsymbol{x}_k$, $\boldsymbol{u}_k$, and $\boldsymbol{z}_k$ are state variable, input, and noisy output measurements, respectively. Also, the vectors valued $\boldsymbol{f}$ and $\boldsymbol{h}$ functions are usually assumed to be differentiable but otherwise arbitrary. $\boldsymbol{f}$ is a function that relates the recent values of input and state to the current input and $\boldsymbol{h}$ is a function that relates current input and state values to the current measurement. In tracking literature, Equation (1) is usually called state process and Equation (2) is referred to as measurement process. The state vector evolves according to a generally nonlinear but stationary Markov dynamics driven by the input and by the noise source. The outputs $\boldsymbol{z}_k$ are nonlinear, noisy but stationary and instantaneous functions of the current state and current input.

In cases which the model parameters including $\boldsymbol{f}$, $\boldsymbol{h}$ and noise processes are assumed to be known, the goal is to develop an algorithm to estimate the state evolution using the noisy observation trajectory. For instance, Kalman filter[8] is the optimum solution to the case where $\boldsymbol{f}$ and $\boldsymbol{h}$ are linear functions and noise processes are Gaussian. An another example EKF[8] provides a sub-optimal solution to the nonlinear case. In both of these methods, it is assumed that all information about the state and measurements are confined in second order statistics, i.e. mean and covariance matrix and therefore they provide simple update equations for mean and covariance matrices as general solution to the estimation problem. Motivated by the fact that the signal probability density function (*pdf*) embodies all the information about the signal, Bayesian approach provides optimum algorithms that exploit *pdf*'s rather than moments of signal in their calculations. These algorithms are solutions to the general nonlinear and non-Gaussian state estimation problem. Particle filters,[10] [11] are examples of the algorithms in this category.

However, there are cases which the model parameters are not known or are known with a degree of uncertainty. Unknown or uncertain noise statistics, uncertain state transition function $\boldsymbol{f}$, unknown moving object scenario and damaged and biased measurement devices are examples in tracking. In these scenarios, standard estimation algorithms which assume perfect knowledge of the model parameters are not accurate anymore.

Therefore the main idea is to estimate the state evolution using unknown or uncertain model parameters. The difficulty is that both state evolution trajectory and model parameters are unknown. The classical approach to solving this problem is to treat the parameters as extra state variables and to apply an extended Kalman filtering algorithm to the nonlinear system with the state vector augmented by the parameters.[9] In contrast, the algorithm presented in this paper is a iterative method for jointly identification of the model parameters and state estimation.

3. EXPECTATION MAXIMIZATION

When a training set consisting state and measurement data is available, the straightforward solution to the parameters identification is to the maximum likelihood estimation. The estimation is resulted to a simple least square solution when the noise is Gaussian. However, in our case where the state is also unknown the likelihood function is not completely know. The expectation maximization (EM)[1] algorithm provides an iterative method for solving the maximum likelihood estimation problem when the complete knowledge of likelihood function is not available. The algorithm is developed based on the simple idea that since the likelihood function is not completely known, the likelihood function can be replaced with the its expectation over all possible values of unknown variable, e.g. states. Therefore the main goal is to integrate over uncertain estimates of the unknown hidden states and optimize the resulting marginal likelihood of the parameters given the observed data.

Observe that the likelihood of measurements, also known as *partial likelihood*, can be considered as the marginal distribution of complete-data likelihood over the unknown states. Therefore, in the E-step assuming that the model parameters are known, the expectation of partial likelihood of observed data is computed. This expectation provides the expected partial likelihood function also referred to as *Auxiliary function*.[1] In M-step the new model parameters are calculated by maximizing the auxiliary function. This iterations continue until the convergence. The likelihood of observations is defined as:

$$\mathcal{L}(\theta) = \log p(\boldsymbol{z}|\boldsymbol{u}, \theta) \tag{3}$$

where $\boldsymbol{z} = \{z_1, ..., z_K\}$ is the entire sequence of observed measurements, $\boldsymbol{x} = \{x_1, ..., x_K\}$ is the state variables, $\boldsymbol{u} = \{u_1, ..., u_K\}$ is input sequence, and θ is the vector of parameters.

Maximizing the likelihood function as a function of θ is equivalent to maximizing the log-partial likelihood:

$$\mathcal{L}(\theta) = \log p(\boldsymbol{z}|\boldsymbol{u}, \theta) = \log \int_{\boldsymbol{x}} p(\boldsymbol{x}, \boldsymbol{z}|\boldsymbol{u}, \theta) d\boldsymbol{x} \tag{4}$$

Obviously, maximizing this function is very complex as the function should be optimized over all possible values that state variable $\boldsymbol{x}$ can assume. Instead, by defining a variational distribution $U(\boldsymbol{x})$ over the hidden state variables, a lower bound to the partial likelihood is obtained[1]:

$$\begin{aligned}
\mathcal{L}(\theta) &= \log \int_{\boldsymbol{x}} p(\boldsymbol{x}, \boldsymbol{z}|\boldsymbol{u}, \theta) d\boldsymbol{x} && (5)\\
&= \log \int_{\boldsymbol{x}} U(\boldsymbol{x}) \frac{\boldsymbol{p}(\boldsymbol{x}, \boldsymbol{z}|\boldsymbol{u}, \theta)}{U(\boldsymbol{x})} d\boldsymbol{x} && (6)\\
&\geq \int_{\boldsymbol{x}} U(\boldsymbol{x}) \log \frac{p(\boldsymbol{x}, \boldsymbol{z}|\boldsymbol{u}, \theta)}{U(\boldsymbol{x})} d\boldsymbol{x} && (7)\\
&= \mathcal{F}(U, \theta). && (8)
\end{aligned}$$

The EM algorithm alternates between maximizing $\mathcal{F}$ with respect to the distribution $U(\boldsymbol{x})$ and the parameters θ, respectively. Starting from some initial parameters θ_0 the algorithm alternately apply:

$$E-Step: \quad U_{t+1} = \arg\max_{U} \mathcal{F}(U, \theta_t) \tag{9}$$

$$M-Step: \quad \theta_{t+1} = \arg\max_{\theta} \mathcal{F}(U_{t+1}, \theta) \tag{10}$$

The E-step involves with estimating the best distribution of hidden state that makes the expectation of log-likelihood maximum. It is easy to verify that the maximum in the E-step results when $U(\boldsymbol{x})$ is exactly the conditional posterior distribution of $\boldsymbol{x}$:

$$U^*{}_{t+1} = p(\boldsymbol{x}|\boldsymbol{z}, \boldsymbol{u}, \theta_t), \tag{11}$$

Since, at the end of each E-step the likelihood function $\mathcal{F}$ meets the equality, $\mathcal{F}(U^*_{t+1}, \theta_t) = \mathcal{L}(\theta_t)$, and the fact that in M-step the optimization is done over θ, it is guaranteed that the likelihood will not decrease in each iteration.

The M-step involves system identification using the state estimates from the smoother. Therefore, at the heart of EM learning procedure is the following idea: use the solutions to the filtering/smoothing problem to estimate the unknown hidden states given the observations and the current model parameters. Then use this suspicious complete data to solve for for model parameters using maximum likelihood estimation.

4. THE E-STEP, THE MCMC COMPUTATION OF POSTERIOR

Therefore, to solve the general case of E-step it is straightforward to use any method of approximating the posterior *pdf*. The approximated posterior *pdf* provides the smoothed states based on fictitious parameters assumed to be known. This is exactly a smoothing problem in which the hidden states trajectory is estimated given the observation/inputs and the model parameters.

The key idea in Monte Carlo Markov chain (MCMC) methods is to build an ergodic Markov chain whose equilibrium distribution is the desired posterior distribution $p(\boldsymbol{x}|\boldsymbol{z},\boldsymbol{u},\theta)$.[11] Under weak assumptions, the samples generated by the Markov chain are asymptotically distributed according to posterior distribution. The generated samples allow easy evaluation of all posterior features of interest. For instance, for any p-integrable function $f(.)$, $(1/N)\sum_{i=0}^{N-1} w^i f(\boldsymbol{x}^i) \rightarrow \int_{\boldsymbol{x}} f(\boldsymbol{x})p(\boldsymbol{x}|\boldsymbol{z},\boldsymbol{u},\theta)d\boldsymbol{x}$ almost surely, where the weights w^i represent the probability realizations of each sample $\boldsymbol{x}^i$. This will be used in the next section to implement the necessary expectations (22) and (23).

We use one of the first versions of procedures for generating the samples, known as Metropolis-Hastings algorithm.[11] By assuming a proposal density $q(.|\boldsymbol{x})$ and after a burn-in period, the following procedure generates the necessary samples from $\pi = p(\boldsymbol{x}|z,u,\theta)$:

1) Sample $\boldsymbol{x}^* \sim q(\boldsymbol{x}^*|\boldsymbol{x}^{(i-1)})$

2) Evaluate

$$\alpha(\boldsymbol{x}^{(i-1)},\boldsymbol{x}^*) = \min\{1, \frac{\pi(\boldsymbol{x}^*)q(\boldsymbol{x}^{(i-1)}|\boldsymbol{x}^*)}{\pi(\boldsymbol{x}^{(i-1)})q(\boldsymbol{x}^*|\boldsymbol{x}^{(i-1)})}\} \tag{12}$$

3) $\boldsymbol{x}^{(i)} = \boldsymbol{x}^*$ with probability $\alpha(\boldsymbol{x}^{(i-1)},\boldsymbol{x}^*)$; otherwise, $\boldsymbol{x}^{(i)} = \boldsymbol{x}^{(i-1)}$.

In an online version of the algorithm discussed in Section(6.3) the samples generated from an initial MCMC setup can be used as particles in a so-called "*particle filter*"[10] to update the approximated posterior distribution only the last arrived measurement.

5. THE M-STEP

The posterior *pdf* of states given observations obtained in the E-step is replaced with distribution $p(\boldsymbol{x})^*$ in (13) to define the objective function for the M-step:

$$\theta^*{}_{t+1} = \arg\max_{\theta} \int_{\boldsymbol{x}} p(\boldsymbol{x}|\boldsymbol{z},\boldsymbol{u},\theta_t)\log p(\boldsymbol{x},\boldsymbol{z}|\boldsymbol{u},\theta)d\boldsymbol{x} \tag{13}$$

Since $\mathcal{F} = \mathcal{L}$ at the beginning of each M-step, and since the E-step does not change θ, it is guaranteed not to decrease the likelihood after each EM step.

The M-step is in fact a nonlinear regression problem, in which a parameterized function is being fit on the complete-data likelihood to make the maximization feasible. Besides, it is necessary to maximize the objective function over all possible values of hidden states. Therefore, standard regression and curve-fitting methods are usually difficult to apply.

In[9] an approximate method for fitting radial basis function over the gaussian cloud of smoothed states is given. This method is proposed based on this assumption that the E-step is implemented by an EKF and therefore, the estimated posterior is a Gaussian cloud, i.e. modelled by first two moments. This assumption makes the integration and maximization in M-step computationally efficient.

In this paper we extend the algorithm to a more general case in which the noise can be non-Gaussian. In this algorithm a MofG kernels are being fit on complete-data likelihood. The expectation is over the estimated posterior distribution. The maximization results in closed form update equations for model parameters.

5.1. Fitting Mixture of Gaussians to State-Measurement Data

It is assumed that measurement process is uncertain. Therefore, in this section, a MofG is used to approximate the measurement process. Consider the following nonlinear mapping from state vector $\boldsymbol{x}$ to the measurement vector $\boldsymbol{z}$:

$$\boldsymbol{z} = \sum_{i=1}^{L} m_i g_i(\boldsymbol{x}) + A\boldsymbol{x} + \boldsymbol{b} + \boldsymbol{r} \tag{14}$$

where r is a zero-mean Gaussian noise variable with covariance Q. m_i's are scaler valued coefficients of Gaussian kernels g_j with fixed center $\boldsymbol{c}_i$ and the fixed width given by the covariance matrix S_i:

$$g_i(\boldsymbol{x}) = |2\pi S_i|^{-1/2} \exp[-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{c}_i)^T S_i^{-1}(\boldsymbol{x} - \boldsymbol{c}_i)] \tag{15}$$

The mapping is used to represent and approximate the measurement process Equation (2) by the following assignments, respectively:

$$\boldsymbol{x} \leftarrow \boldsymbol{x_t}, \boldsymbol{z} \leftarrow z_t \tag{16}$$

The parameters of the mixture include L coefficients m_i, the matrix A multiplying state variable $\boldsymbol{x}$, and the input bias vector $\boldsymbol{b}$, and the noise covariance Q. The goal is to fit this Gaussian mixtures to data $(\boldsymbol{z}, \boldsymbol{x})$.

Let $\hat{\boldsymbol{z}}_\theta(\boldsymbol{x}, \boldsymbol{u}) = \sum_{i=1}^{L} m_i g_i(\boldsymbol{x}) + A\boldsymbol{x} + \boldsymbol{b}$, where θ is the set of parameters. The log-likelihood of single fully observed data point under the model would be:

$$-\frac{1}{2}[\boldsymbol{z} - \hat{\boldsymbol{z}}_\theta(\boldsymbol{x})]^T Q^{-1}[\boldsymbol{z} - \hat{\boldsymbol{z}}_\theta(\boldsymbol{x})] - \frac{1}{2} ln|Q| + constant. \tag{17}$$

By substituting the approximated log-likelihood in Equation (13) and since the values for $(\boldsymbol{z}, \boldsymbol{x})$ are uncertain, the M-step optimization would be:

$$\min_{\theta, Q}\{\int_{\boldsymbol{x}}\int_{\boldsymbol{z}} p(\boldsymbol{x}|\boldsymbol{z}, \theta_t)[\boldsymbol{z} - \hat{\boldsymbol{z}}_\theta(\boldsymbol{x})]^T Q^{-1}[\boldsymbol{z} - \hat{\boldsymbol{z}}_\theta(\boldsymbol{x}, \boldsymbol{u})] d\boldsymbol{x} d\boldsymbol{z} + \ln|Q|\}. \tag{18}$$

Defining:

$$\theta = [m_1, m_2, ..., m_L, A, b], \tag{19}$$

$$\Phi = [g_1(\boldsymbol{x}), g_2(\boldsymbol{x}), ..., g_L(\boldsymbol{x}), \boldsymbol{x}^T, \boldsymbol{u}_T, 1]^T, \tag{20}$$

and denoting the expectation over posterior distribution, $p(\boldsymbol{x}|\boldsymbol{z}, \theta_t)$ by $\langle\rangle$ the objective is written as:

$$\min_{\theta, Q}\{\langle[\boldsymbol{z} - \theta\Phi]^T Q^{-1}[\boldsymbol{z} - \theta\Phi]\rangle + \ln|Q|\}. \tag{21}$$

Taking derivatives with respect to θ and ϕ, and setting the results to zero and solving for them gives:

$$\hat{\theta} = \langle\boldsymbol{z}\Phi^T\rangle\langle\Phi\Phi^T\rangle^{-1} \tag{22}$$

$$\hat{Q} = \langle\boldsymbol{z}\boldsymbol{z}^T\rangle - \hat{\theta}\langle\Phi\boldsymbol{z}^T\rangle. \tag{23}$$

In other words, given the expectations in the angular brackets, the optimal parameters can be solved for via a set of linear equations. Besides, as we have approximated the posterior by a finite number of points in E-step, the expectation will be simplified to summations over a finite points.

6. A BEARING-ONLY TARGET TRACKING PROBLEM

The success of the presented method is evaluated by solving a bearing-only target tracking problem. The problem consists of a linear state evolution and a nonlinear measurement process. For defining the problem the exact definition as given in,[12] [8] in followed. In the simulated scenario a platform with a sensor moves according to the discrete time equations:

$$\boldsymbol{x}_k^p = \overline{\boldsymbol{x}}_k^p + \Delta \boldsymbol{x}_k^p \qquad k = 0, 1, ..., 20 \tag{24}$$

$$\boldsymbol{y}_k^p = \overline{\boldsymbol{y}}_k^p + \Delta \boldsymbol{y}_k^p \qquad k = 0, 1, ..., 20 \tag{25}$$

where the $\overline{\boldsymbol{x}}_k^p$ and $\overline{\boldsymbol{y}}_k^p$ are the average platform position coordinates and the perturbations $\Delta \boldsymbol{x}_k^p$ and $\Delta \boldsymbol{y}_k^p$ are assumed to be mutually independent zero-mean Gaussian white noise sequences with variances $r_x = 1m^2$ and $r_y = 1m^2$, respectively.

The average (unperturbed) platform motion is assumed to be horizontal with constant velocity. Its position as a function of the discrete time k (in meters) is:

$$\hat{\boldsymbol{x}}_k^p = 4k \tag{26}$$

$$\hat{\boldsymbol{y}}_k^p = 20 \tag{27}$$

A target moves on the $\boldsymbol{x}$-axis according to

$$\boldsymbol{x}_{k+1} = F_k \boldsymbol{x}_k + K \boldsymbol{v}_k \tag{28}$$

where:

$$\boldsymbol{x}_k = \begin{bmatrix} x_{1k} \\ x_{2k} \end{bmatrix} \tag{29}$$

$$F_k = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \tag{30}$$

$$K = \begin{bmatrix} T^2/2 \\ T \end{bmatrix} \tag{31}$$

with x_1 denoting the position (in meters) and x_2 denoting the velocity (in m/s) on the target, $T = 1s$ is the sampling period, and the $\boldsymbol{v}_k$ is zero mean white process noise with variance $q = 10^2 m^2/s^4$.

The initial condition is:

$$\boldsymbol{x}_0 = \begin{bmatrix} 80 \\ 1 \end{bmatrix} \tag{32}$$

The sensor measurement is

$$z_k^m = z_k + w_k^s \tag{33}$$

where

$$z_k = \boldsymbol{h}(\boldsymbol{x}_k^p, \boldsymbol{y}_k^p, \boldsymbol{x}_{1k}) = \tan^{-1} \frac{\boldsymbol{y}_k^p}{\boldsymbol{x}_{1k} - \boldsymbol{x}_k^p} \tag{34}$$

is the bearing between the horizontal and the line of sight from the sensor to the target, and the sensor noise w_k^s is zero mean white Gaussian with variance $r^s = (3°)^2$.The sensor noise is assumed independent of the sensor platform perturbations.

The estimation of the target's state is to be done using the measurements (**??**) and the knowledge of the unperturbed platform location (26) and (27). However, the platform location perturbations induce additional errors in the measurements. The effects of these errors is evaluated by expanding the nonlinear measurement function $\boldsymbol{h}$ about the average platform position. The resulting measurement evolution can be written as:

$$z_k = \boldsymbol{h}(\overline{\boldsymbol{x}}_k^p, \overline{\boldsymbol{y}}_k^p, \boldsymbol{x}_{1k}) + w_k = \tan^{-1} \frac{\overline{\boldsymbol{y}}_k^p}{\boldsymbol{x}_{1k} - \overline{\boldsymbol{x}}_k^p} + w_k, \tag{35}$$

where the equivalent measurement noise w_k is a zero mean white Gaussian with variance:

$$E[w(k)^2] = r_k = \frac{(\overline{\boldsymbol{y}}_k^p)^2 r_x + (\boldsymbol{x}_{1k} - \overline{\boldsymbol{x}}_k^p)^2 r_y}{\{(\overline{\boldsymbol{y}}_k^p)^2 + (\boldsymbol{x}_{1k} - \overline{\boldsymbol{x}}_k^p)^2\}^2} + r_k^s \tag{36}$$

Notice that the variance of the equivalent measurement noise is time varying. For more details on modelling the new measurement process refer to.[8] The state $\boldsymbol{x}$ is to be estimated based on the measurement equation (35) with the knowledge of only the average platform motion (26) and (27).

6.1. The Extended Kalman Filter (EKF)

When the model parameters are known, one method for solving the problem is EKF in which the nonlinear measurement function is linearized and the problem solved using a Kalman filter. We summarize the implementation of EKF in the following equations.[12]

State Prediction:

$$\hat{\boldsymbol{x}}_{k+1|k} = F_k \hat{\boldsymbol{x}}_{k|k} \tag{37}$$

Covariance matrix of state prediction:

$$P_{k+1|k} = F_k P_{k|k} F_k^T + Q_k \tag{38}$$

Measurement matrix (linearized):

$$H_{k+1} = \partial \boldsymbol{h} / \partial \boldsymbol{x}|_{\boldsymbol{x} = \hat{\boldsymbol{x}}_{k+1|k}} = \begin{bmatrix} \frac{\hat{y}^p}{(\hat{x}_1 - \overline{x})^2 + (\hat{y}^p)^2} & 0 \end{bmatrix} \tag{39}$$

Measurement residual:

$$\nu_{k+1} = z_{k+1}^m - \tan^{-1} \frac{\overline{y}^p}{\hat{\boldsymbol{x}}_1 - \overline{\boldsymbol{x}}^p} \tag{40}$$

Equivalent measurement noise covariance: Measurement residual:

$$R_{k+1} = \frac{(\overline{\boldsymbol{y}}^p)^2 r_x + (\hat{\boldsymbol{x}}_1 - \overline{\boldsymbol{x}}^p)^2 + (\overline{\boldsymbol{x}}^p)^2 r_y}{[(\overline{\boldsymbol{y}}^p)^2 + (\hat{\boldsymbol{x}}_1 - \overline{\boldsymbol{x}}^p)^2 + (\overline{\boldsymbol{x}}^p)^2]^2} + r^s \tag{41}$$

Predicted measurement covariance:

$$S_{k+1} = H_{k+1} P_{k+1|k} H_{k+1}^T + R_{k+1} \tag{42}$$

Gain matrix computation:

$$K_{k+1} = P_{k+1|k} H_{k+1}^T S_{k+1} \tag{43}$$

Updated state:

$$\boldsymbol{x}_{k+1|k+1} = \boldsymbol{x}_{k+1|k} + K_{k+1} \nu_{k+1} \tag{44}$$

Updated state covariance matrix:

$$P_{k+1|k+1} = (I - K_{k+1} H_{k+1}) P_{k+1|k} \tag{45}$$

6.2. Online Algorithm

The main drawback of EM-type algorithms is that they rely of the knowledge of all data sequence and therefore are *batch* algorithm. However, in case that partial maximization of the likelihood is tolerated, it is straightforward to derive the online version of the algorithm. The main idea is that in E-step instead of smoothing all the state data, only the last state is smoothed. In addition, in M-step, instead of maximizing the likelihood, increasing it towards its maximum. Since the objective function in M-step (21) is quadratic with respect to the unknown parameters θ, partial maximization in M-step results in an RLS-type algorithm.

6.3. Online E-step, Particle Filter

In complete batch E-step, it is assumed that all of the state data are smoothed and therefore the posterior distribution is updated in each E-step for all the previous state data. However, i a partial E-step, one can only smooth the last single state data. It is similar to one-step smoothing which makes the E-step much faster with the price of less accuracy in the parameters estimation within the following M-steps.

Additionally, within the MCMC simulation the posterior distribution $p(\boldsymbol{x}|z, \boldsymbol{u}, \theta)$ is estimated using all the states data up to the current time. However, in an online regime, it is possible to update the distribution with only the last received measurement data. Interested readers are referred to.[10] In this method, the samples initially generated by an MCMC method are associated with a number of particles $\boldsymbol{x}^i \quad i = 1, ..., N$ with their corresponding weights $w^i \quad i = 1, ..., N$. Then, the weight w_k^i for particle i at time instance k is updated according to the following equations:

$$w_k^i \propto w_{k-1}^i \frac{p(z_k^m|\boldsymbol{x}_k^i, \Delta\boldsymbol{x}_k^p, \Delta\boldsymbol{y}_k^p, \theta)p(\boldsymbol{x}_k^i|\boldsymbol{x}_{k-1}^i)}{r(\boldsymbol{x}_k^i|\boldsymbol{x}_{k-1}^i, z_k)}, \tag{46}$$

where $r(.)$ is usually called *importance density*.[11] Other necessary entities are defined as:

$$\boldsymbol{x}_k^i \sim p(\boldsymbol{x}_{k-1}|\boldsymbol{x}_k^i), \tag{47}$$

$$\Delta\boldsymbol{x}_k^{pi} \sim p(\Delta\boldsymbol{x}_k^p), \tag{48}$$

$$\Delta\boldsymbol{y}_k^{pi} \sim p(\Delta\boldsymbol{y}_k^p), \tag{49}$$

The particles and their weights can be used to reconstruct the posterior density:

$$p(\boldsymbol{x}_k^i|z_k^m, \Delta\boldsymbol{x}_k^{pi}, \Delta\boldsymbol{y}_k^{pi}, \theta) \approx \sum_{i=1}^{N} w_k^i \delta(\boldsymbol{x}_k - \boldsymbol{x}_k^i). \tag{50}$$

Therefore the posterior distribution data is updated using the knowledge of estimated posterior from the last E-step and only the last measurement data.

The problem with particle filter is that the weight are becoming overly skewed and all but a very few of the particles have negligible weights. This problem prevents the contribution of all the particles in approximating the distribution. There are a number of proposed techniques called *resampling* to avoid this problem. A simple minimum variance scheme firstly proposed by Kitagawa[13] and used in[12] for tracking application can be used in this paper. This resampling technique probabilistically replicates particles with large weights and discards particles with small weights.

6.4. Online M-step, RLS Algorithm

Since the objective function in M-step (21) is quadratic with respect to the unknown parameters θ, partial maximization in M-step results in an RLS-type algorithm. The speed for a partial M-step is gained with the price of increasing the likelihood function instead of maximizing it.

The main idea is to develop recursive update equations for estimated parameters (22) and (23). Firstly, using the *matrix inversion lemma* a recursive estimate of $\langle \Phi\Phi^T \rangle^{-1}$ is derived:

$$P_k = P_{k-1} - \frac{P_{k-1}\langle \Phi\Phi^T \rangle_k P_{k-1}}{1 + \langle \Phi^T P_{k-1} \Phi \rangle_k}, \tag{51}$$

where k is the index of time in and $\langle . \rangle_k$ is expectation evaluated at $(\boldsymbol{x}_k, z_k)$.

After some mathematical manipulations recursive update equations for the parameters can be written as follows:

$$\theta_k = \theta_{k-1} + (\langle z\Phi \rangle_k - \theta_{k-1}\langle \Phi\Phi^T \rangle_k)P_k, \tag{52}$$

$$Q_k = Q_{k-1} + \frac{1}{k}[\langle z^2 \rangle_k - \theta_k \langle \Phi z \rangle_k - Q_{k-1}] \tag{53}$$

6.5. The Algorithm

We assume that the measurement process model is known with a degree of uncertainty. Therefore it is necessary to identify the measurement process while tracking the target. For this purpose, a nonlinear regression method is used to identify the nonlinear measurement process. The proposed algorithm consist of two consecutive cycles. In E-step MCMC method is used to approximate the posterior *pdf* and initialize a particle filter. The a particle filter is used to update the posterior with the last measurement. In M-step a regression method is used to fit a MofG on smoothed states to identify the measurement process model. The proposed algorithm has the following steps.

- MofG Initialization: The center of Gaussian kernels in MofG are initialized over grid points formed on state space. The initial parameters if mixture are trained using the data generated by uncertain measurement process. The uncertainty is generated by randomly adding or subtracting a 10% offset to the nonlinear process.

- E-step: The track is initialized using M-H MCMC method, Section (4), to estimate the initial value for the state. The number of particles for MCMC method is assumed to be 10,000. Then the posterior distribution is estimated using a particle filter, Section (6.3).

  According to Baye's rule, the posterior distribution is defined as:

  $$P(\boldsymbol{x}_k|z_k, \boldsymbol{u}_k, v_k) = \frac{P(z_k|\boldsymbol{x_k}, \boldsymbol{u}_k, v_k)P(\boldsymbol{x}_k)}{\int_{\boldsymbol{x}} P(z_k|\boldsymbol{x_k}, \boldsymbol{u}_k, v_k)P(\boldsymbol{x}_k)d\boldsymbol{x}} \tag{54}$$

  Besides, from (2) for the likelihood function we have:

  $$P(z_k|\boldsymbol{x_k}, \boldsymbol{u}_k, v_k) = \mathcal{N}(z_k - \tan^{-1}\frac{\overline{\boldsymbol{y}}_k^p}{\boldsymbol{x}_{1k} - \overline{\boldsymbol{x}}_k^p}, r_k). \tag{55}$$

  Furthermore, the prior distribution $P(\boldsymbol{x}_(k+1)|\boldsymbol{x}_(k))$ is obtained by the inherent state evolution process (28):

  $$p(\boldsymbol{x}^{i+1}|\boldsymbol{x}^{(i)}) = \mathcal{N}(\boldsymbol{x}^* - \boldsymbol{x}^{(i)}, R) \tag{56}$$

  By substituting these equations into (54), the necessary equations for M-H MCMC (12) and the particle filter (Section 6.3) are obtained.

- M-step: MofG fitting

  The estimated posterior which consists of a large number of particles is used to evaluate the necessary expectations using the following general relationship:

$$\langle f(\boldsymbol{x})\rangle = \int_{\boldsymbol{x}} f(\boldsymbol{x}) p(\boldsymbol{x}_k|z_k, \boldsymbol{u}_k, v_k) d\boldsymbol{x} = (1/N) \sum_{i=0}^{N-1} w^i f(\boldsymbol{x}^i) \tag{57}$$

  where w^i is the posterior probability evaluated at $\boldsymbol{x}^i$. Using this general equation, the parameters of MofG, i.e. the parameter vector θ and model noise covariance Q, are estimated by using (22) and (23). The estimated parameters are used in the next E-step.

- Continue until the convergence.

7. SIMULATION RESULTS

The proposed method is applied to solve the bearing-only problem with uncertain model parameters. It is assumed that the measurement devices have 10% bias so that the measurements are uncertain up to 10%. Therefore the nonlinear dynamics of measurement process, Equation (2), is assumed to be uncertain and therefore can not be used directly in estimation process.

The standard filtering methods like EKF are not able to estimate the states when the measurement process is uncertain. To show this EKF is used to solve the problem when the measurement process is known with 10% uncertainty. It can be seen from Figure(**??**) that the EKF looses the track in the very instances of time.

The main idea in applying the proposed method is to improve the estimation by iteratively approximating a model for the nonlinear measurement process using the smoothed data (M-step) and then smoothing the states using the new approximated model (E-step).

In M-step a MofG is fitted on the estimated state data obtained in the E-step. However, for the first iteration when there is no data available, the parameters of the MofG should be initialized. Therefore, the center of Gaussian kernels are initialized on a uniform grid formed in state space. Then the MofG is trained using the uncertain data acquired from the uncertain measurement process. By this, the MofG is fitted on the uncertain data generated by the uncertain measurement process. The parameters estimated by this initialization is used to approximate the nonlinear measurement process in the first E-step.

In the E-step the conditional posterior *pdf* of the states given the all previous measurements are estimated. The tracking initialization in which the *pdf* of the posterior is generated is done by M-H MCMC algorithm presented in Section (4). The *proposal function* is a Gaussian with mean equal to the estimated state in the previous time instance and the variance is chosen so that the support of the function covers adequate space around the current state. Then a particle filter presented in Section (6.3) is used to update the generated posterior *pdf* and estimate the states. The number of particles are assumed to be 10,000. The prior *pdf* for state variables is assumed based on the state evolution process (1). Finally, the estimated *pdf* is used to estimate the states as conditional mean, that provides the estimated states.

The estimated states are used in the M-step to re-identify the MofG parameters via Equations (22) and (23). The necessary expectations are computed by means of estimated posterior *pdf* evaluated in the E-step. The algorithm continues until the convergence.

In the first simulations it is assumed that the measurement process is known completely and the MofG is trained using the data generated from this process. The RMS error for 50 Monte Carlo runs of E-step is given in Figure (**??**). It can be seen that the algorithm ability of tracking the states is comparable to a completely known model EKF. Also, the RMS error for tracking the position for 50 Monte Carlo runs (for E-steps) of the algorithm with uncertain measurement process is given. It can be seen that the proposed algorithm is able to manage the unknown dynamics by identifying it and use this approximation for the smoothing the states.

Although the simulation results are provided for a Gaussian nonlinear measurement dynamics, the application of MCMC method for estimating the posterior *pdf* enables the method to work for a Non-Gaussian noise model. The cost for this ability is a comparatively expensive amount of computations necessary for each E-step.

8. CONCLUSIONS

An EM-type algorithm for solving a joint estimation-identification of generally nonlinear Non-Gaussian state-space tracking problem is proposed. The proposed method is a generalization of original work by Reweis[9] that has the credit for using the EM method for solving the general state-space problem.

The method was applied to solve a bearing-only tracking problem in which the nonlinear measurement dynamics were assumed to be biased up to 10%. The simulations results showed that the algorithm was able to both identify and approximate the measurement dynamics and to estimate the states successfully.

In proposed method it is assumed that the Gaussian kernels have fixed centers and widths. It is straightforward to relax this assumptions to have a variable MofG in regression method used in M-step.

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