Derivation of CRLB for Wideband Array Signal Processing Using MCMC Methods

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I. Derivation of CRLB

Let $\boldsymbol{\theta}$ be the parameter vector defined as follows:

$$\boldsymbol{\theta} \triangleq \{\tau_k, \boldsymbol{s}_k(n)\}, k = 0, ..., K-1, n = 1, ..., N, \tag{1}$$

and $l(\boldsymbol{\theta})$ be the likelihood function, which is defined as:

$$l(\boldsymbol{\theta}) = p(\boldsymbol{Y}|\boldsymbol{\theta}), \qquad \boldsymbol{Y} = \{\boldsymbol{y}(n), n = 1, ..., N\}, \tag{2}$$

$$= \prod_{n=1}^{N} \frac{1}{(2\pi\sigma_w^2)^{M/2}} \exp\left\{ \frac{-1}{2\sigma_w^2} \left(\boldsymbol{y}(n) - \sum_{l=0}^{L-1} \tilde{\boldsymbol{H}}_l(\boldsymbol{\tau}) \boldsymbol{\mathcal{S}}(n-l) \right)^T \left(\boldsymbol{y}(n) - \sum_{l=0}^{L-1} \tilde{\boldsymbol{H}}_l(\boldsymbol{\tau}) \boldsymbol{\mathcal{S}}(n-l) \right) \right\}, \tag{3}$$

where

$$\tilde{\boldsymbol{H}}_l(\boldsymbol{\tau}) \triangleq \left[\tilde{\boldsymbol{H}}_l(\tau_0), \tilde{\boldsymbol{H}}_l(\tau_1), \ldots, \tilde{\boldsymbol{H}}_l(\tau_{K-1}) \right], \qquad \tilde{\boldsymbol{H}}_l(\tau_k) \in \mathcal{R}^{M\times 1}, \tag{4}$$

and

$$\boldsymbol{\mathcal{S}}(n-l) \triangleq [s_0(n), s_1(n), \ldots, s_{K-1}(n)]^T, \qquad \boldsymbol{\mathcal{S}}(n-l) \in \mathcal{R}^{K\times 1}. \tag{5}$$

Defining $\boldsymbol{u}(n)$ as follows

$$\boldsymbol{u}(n) \triangleq \boldsymbol{y}(n) - \sum_{l=0}^{L-1} \tilde{\boldsymbol{H}}_l(\boldsymbol{\tau}) \boldsymbol{\mathcal{S}}(n-l) \tag{6}$$

and taking the natural logarithm of (3) yields

$$L(\boldsymbol{\theta}) \triangleq \ln[l(\boldsymbol{\theta})], \tag{7}$$

$$= -\frac{1}{2\sigma_w^2} \sum_{n=1}^{N} \boldsymbol{u}^T(n)\boldsymbol{u}(n) - \frac{MN}{2} \ln\left(2\pi\sigma_w^2\right). \tag{8}$$

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A. *Derivation of* $\frac{\partial L(\boldsymbol{\theta})}{\partial \tau_k}$

$$
\begin{aligned}
\frac{\partial L(\boldsymbol{\theta})}{\partial \tau_k} &= -\frac{1}{2\sigma_w^2}\left\{\frac{\partial}{\partial \tau_k}\sum_{n=1}^{N}\boldsymbol{u}^T(n)\boldsymbol{u}(n)\right\}, && (9)\\
&= -\frac{1}{2\sigma_w^2}\left\{\sum_{n=1}^{N}\left(\frac{\partial \boldsymbol{u}(n)}{\partial \tau_k}\right)^T\boldsymbol{u}(n)+\sum_{n=1}^{N}\boldsymbol{u}^T(n)\left(\frac{\partial \boldsymbol{u}(n)}{\partial \tau_k}\right)\right\}, && (10)
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{\partial \boldsymbol{u}(n)}{\partial \tau_k} &= \frac{\partial}{\partial \tau_k}\left(\boldsymbol{y}(n)-\sum_{l=0}^{L-1}\tilde{\boldsymbol{H}}_l(\boldsymbol{\tau})\boldsymbol{\mathcal{S}}(n-l)\right), && (11)\\
&= -\sum_{l=0}^{L-1}\frac{\partial \tilde{\boldsymbol{H}}_l(\boldsymbol{\tau})}{\partial \tau_k}\boldsymbol{\mathcal{S}}(n-l), && (12)\\
&= -\sum_{l=0}^{L-1}\frac{\partial}{\partial \tau_k}\left[\tilde{\boldsymbol{H}}_l(\tau_0),\ldots,\tilde{\boldsymbol{H}}_l(\tau_k),\ldots,\tilde{\boldsymbol{H}}_l(\tau_{K-1})\right]\begin{bmatrix} s_0(n-l)\\ \vdots \\ s_k(n-l)\\ \vdots \\ s_{K-1}(n-l)\end{bmatrix}, && (13)\\
&= -\sum_{l=0}^{L-1}\tilde{\boldsymbol{H}}_l^{'}(\tau_k)s_k(n-l), && (14)\\
&= -\tilde{\boldsymbol{H}}^{'}(\tau_k)\boldsymbol{s}_{\boldsymbol{k}}(n), && (15)
\end{aligned}
$$

where

$$
\tilde{\boldsymbol{H}}_l^{'}(\tau_k) \triangleq \frac{\partial \tilde{\boldsymbol{H}}_l(\tau_k)}{\partial \tau_k}, \qquad (16)
$$

$$
\tilde{\boldsymbol{H}}^{'}(\tau_k) = \left[\tilde{\boldsymbol{H}}_0^{'}(\tau_k),\tilde{\boldsymbol{H}}_1^{'}(\tau_k),\ldots,\tilde{\boldsymbol{H}}_{L-1}^{'}(\tau_k)\right], \qquad (17)
$$

$\boldsymbol{s}_k(n)$ is defined as follows:

$$
\boldsymbol{s}_k(n) = [s_k(n), s_k(n-1),\ldots,s_k(n-L+1)]^T \qquad (18)
$$

and $\mathbf{0}\in\mathcal{R}^{M\times 1}$ is a column vector of zeros. Accordingly, (10) becomes:

$$
\begin{aligned}
\frac{\partial L(\boldsymbol{\theta})}{\partial \tau_k} &= -\frac{1}{2\sigma_w^2}\left\{\sum_{n=1}^{N}\left(\frac{\partial \boldsymbol{u}(n)}{\partial \tau_k}\right)^T\boldsymbol{u}(n)+\sum_{n=1}^{N}\boldsymbol{u}^T(n)\left(\frac{\partial \boldsymbol{u}(n)}{\partial \tau_k}\right)\right\}, && (19)\\
&= -\frac{1}{\sigma_w^2}\sum_{n=1}^{N}\left(-\tilde{\boldsymbol{H}}^{'}(\tau_k)\boldsymbol{s}_{\boldsymbol{k}}(n)\right)^T\boldsymbol{u}(n), && (20)\\
&= \frac{1}{\sigma_w^2}\sum_{n=1}^{N}\boldsymbol{s}_k^T(n)\boldsymbol{g}_k(n), && (21)
\end{aligned}
$$

where

$$\boldsymbol{g}_k(n) \triangleq \left(\tilde{\boldsymbol{H}}^{'}(\tau_k)\right)^T \boldsymbol{u}(n). \tag{22}$$

B. Derivation of $\frac{\partial L(\boldsymbol{\theta})}{\partial s_k(n)}$

$$\begin{aligned}
\frac{\partial L(\boldsymbol{\theta})}{\partial s_k(n)} &= -\frac{1}{2\sigma_w^2}\left\{\frac{\partial}{\partial s_k(n)}\sum_{n=1}^{N}\boldsymbol{u}^T(n)\boldsymbol{u}(n)\right\}, &(23)\\
&= -\frac{1}{2\sigma_w^2}\left\{\sum_{n=1}^{N}\left(\frac{\partial \boldsymbol{u}(n)}{\partial s_k(n)}\right)^T\boldsymbol{u}(n)+\sum_{n=1}^{N}\boldsymbol{u}^T(n)\left(\frac{\partial \boldsymbol{u}(n)}{\partial s_k(n)}\right)\right\}, &(24)
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial \boldsymbol{u}(n)}{\partial s_k(n)} &= \frac{\partial}{\partial s_k(n)}\left(\boldsymbol{y}(n)-\sum_{l=0}^{L-1}\tilde{\boldsymbol{H}}_l(\boldsymbol{\tau})\boldsymbol{\mathcal{S}}(n-l)\right), &(25)\\
&= -\frac{\partial}{\partial s_k(n)}\left\{\tilde{\boldsymbol{H}}_0(\boldsymbol{\tau})\boldsymbol{\mathcal{S}}(n)+\sum_{l=1}^{L-1}\tilde{\boldsymbol{H}}_l(\boldsymbol{\tau})\boldsymbol{\mathcal{S}}(n-l)\right\}, &(26)\\
&= -\frac{\partial}{\partial s_k(n)}\left\{\tilde{\boldsymbol{H}}_0(\boldsymbol{\tau})\boldsymbol{\mathcal{S}}(n)\right\}, &(27)\\
&= -1\times\left[\tilde{\boldsymbol{H}}_0(\tau_0),\ldots,\tilde{\boldsymbol{H}}_0(\tau_k),\ldots,\tilde{\boldsymbol{H}}_0(\tau_{K-1})\right]\frac{\partial}{\partial s_k(n)}\left\{\begin{bmatrix} s_0(n)\\ \vdots \\ s_k(n) \\ \vdots \\ s_{K-1}(n)\end{bmatrix}\right\}, &(28)\\
&= -\tilde{\boldsymbol{H}}_0(\tau_k). &(29)
\end{aligned}$$

Substituting (29) into (24), we have:

$$\begin{aligned}
\frac{\partial L(\boldsymbol{\theta})}{\partial s_k(n)} &= -\frac{1}{2\sigma_w^2}\left\{\sum_{n=1}^{N}\left(\frac{\partial \boldsymbol{u}(n)}{\partial s_k(n)}\right)^T\boldsymbol{u}(n)+\sum_{n=1}^{N}\boldsymbol{u}^T(n)\left(\frac{\partial \boldsymbol{u}(n)}{\partial s_k(n)}\right)\right\}, &(30)\\
&= \frac{1}{\sigma_w^2}\sum_{n=1}^{N}\tilde{\boldsymbol{H}}_0^T(\tau_k)\boldsymbol{u}(n). &(31)
\end{aligned}$$

C. Derivation of $\frac{\partial^2 L(\boldsymbol{\theta})}{\partial \tau_k \partial \tau_p}$

1. If $k = p$

$$\frac{\partial^2 L(\boldsymbol{\theta})}{\partial \tau_k^2} = \frac{1}{\sigma_w^2} \frac{\partial}{\partial \tau_k} \sum_{n=1}^{N} \boldsymbol{s}_k^T(n) \boldsymbol{g}_k(n), \tag{32}$$

$$= \frac{1}{\sigma_w^2} \sum_{n=1}^{N} \boldsymbol{s}_k^T(n) \left(\frac{\partial \boldsymbol{g}_k(n)}{\partial \tau_k} \right), \tag{33}$$

$$= \frac{1}{\sigma_w^2} \sum_{n=1}^{N} \sum_{l=0}^{L-1} \frac{\partial g_k^l(n)}{\partial \tau_k} s_k(n-l), \tag{34}$$

where

$$\frac{\partial g_k^l(n)}{\partial \tau_k} = \left(\frac{\partial \tilde{\boldsymbol{H}}_l^{'}(\tau_k)}{\partial \tau_k} \right)^T \boldsymbol{u}(n) + \left(\tilde{\boldsymbol{H}}_l^{'}(\tau_k) \right)^T \frac{\partial \boldsymbol{u}(n)}{\partial \tau_k}, \tag{35}$$

$$= \left(\tilde{\boldsymbol{H}}_l^{''}(\tau_k) \right)^T \boldsymbol{u}(n) - \left(\tilde{\boldsymbol{H}}_l^{'}(\tau_k) \right)^T \tilde{\boldsymbol{H}}^{'}(\tau_k) \boldsymbol{s}_k(n), \tag{36}$$

where

$$\tilde{\boldsymbol{H}}_l^{''}(\tau_k) \triangleq \frac{\partial^2 \tilde{\boldsymbol{H}}_l(\tau_k)}{\partial \tau_k^2}. \tag{37}$$

Accordingly, (34) becomes:

$$\frac{\partial^2 L(\boldsymbol{\theta})}{\partial \tau_k^2} = \frac{1}{\sigma_w^2} \sum_{n=1}^{N} \sum_{l=0}^{L-1} \left\{ \left(\tilde{\boldsymbol{H}}_l^{''}(\tau_k) \right)^T \boldsymbol{u}(n) - \left(\tilde{\boldsymbol{H}}_l^{'}(\tau_k) \right)^T \tilde{\boldsymbol{H}}^{'}(\tau_k) \boldsymbol{s}_k(n) \right\} s_k(n-l), \tag{38}$$

$$= \frac{1}{\sigma_w^2} \sum_{n=1}^{N} \sum_{l=0}^{L-1} \left(\tilde{\boldsymbol{H}}_l^{''}(\tau_k) \right)^T \boldsymbol{u}(n) s_k(n-l) - \tag{39}$$

$$\frac{1}{\sigma_w^2} \sum_{n=1}^{N} \sum_{l=0}^{L-1} s_k(n-l) \left(\tilde{\boldsymbol{H}}_l^{'}(\tau_k) \right)^T \tilde{\boldsymbol{H}}^{'}(\tau_k) \boldsymbol{s}_k(n), \tag{40}$$

$$= \frac{1}{\sigma_w^2} \sum_{n=1}^{N} \left\{ \boldsymbol{u}^T(n) \tilde{\boldsymbol{H}}^{''}(\tau_k) \boldsymbol{s}_k(n) - \boldsymbol{s}_k(n) \left(\tilde{\boldsymbol{H}}^{'}(\tau_k) \right)^T \tilde{\boldsymbol{H}}^{'}(\tau_k) \boldsymbol{s}_k(n) \right\}, \tag{41}$$

$$= \frac{1}{\sigma_w^2} \left\{ \sum_{n=1}^{N} \boldsymbol{u}^T(n) \tilde{\boldsymbol{H}}^{''}(\tau_k) \boldsymbol{s}_k(n) - \mathrm{tr}\left[\mathcal{H}^{'}(\tau_k, \tau_k) \boldsymbol{R}_{kk}(n) \right] \right\}, \tag{42}$$

where $\mathrm{tr}[\cdot]$ is a trace operator,

$$\mathrm{tr}\left[\mathcal{H}^{'}(\tau_k, \tau_k) \boldsymbol{R}_{kk}(n) \right] \triangleq \sum_{n=1}^{N} \boldsymbol{s}_k(n) \left(\tilde{\boldsymbol{H}}^{'}(\tau_k) \right)^T \tilde{\boldsymbol{H}}^{'}(\tau_k) \boldsymbol{s}_k(n), \tag{43}$$

$$\mathcal{H}^{'}(\tau_k, \tau_k) = \left(\tilde{\boldsymbol{H}}^{'}(\tau_k) \right)^T \tilde{\boldsymbol{H}}^{'}(\tau_k), \tag{44}$$

and

$$\boldsymbol{R}_{kk}(n) = \sum_{n=1}^{N} \boldsymbol{s}_k(n) \boldsymbol{s}_k^T(n). \tag{45}$$

2. If $k \neq p$

$$\frac{\partial^2 L(\boldsymbol{\theta})}{\partial \tau_k \partial \tau_p} = \frac{1}{\sigma_w^2} \frac{\partial}{\partial \tau_p} \sum_{n=1}^{N} \boldsymbol{s}_k^T(n) \boldsymbol{g}_k(n), \quad (46)$$

$$= \frac{1}{\sigma_w^2} \sum_{n=1}^{N} \boldsymbol{s}_k^T(n) \left(\frac{\partial \boldsymbol{g}_k(n)}{\partial \tau_p} \right), \quad (47)$$

$$= \frac{1}{\sigma_w^2} \sum_{n=1}^{N} \sum_{l=0}^{L-1} \frac{\partial g_k^l(n)}{\partial \tau_p} s_k(n-l), \quad (48)$$

where

$$\frac{\partial g_k^l(n)}{\partial \tau_p} = \left(\tilde{\boldsymbol{H}}_l^{'}(\tau_k) \right)^T \frac{\partial \boldsymbol{u}(n)}{\partial \tau_p}, \quad (49)$$

$$= -\left(\tilde{\boldsymbol{H}}_l^{'}(\tau_k) \right)^T \tilde{\boldsymbol{H}}^{'}(\tau_p) \boldsymbol{s}_p(n). \quad (50)$$

Substituting (50) into (48) yields:

$$\frac{\partial^2 L(\boldsymbol{\theta})}{\partial \tau_k \partial \tau_p} = \frac{-1}{\sigma_w^2} \sum_{n=1}^{N} \sum_{l=0}^{L-1} s_k(n-l) \left(\tilde{\boldsymbol{H}}_l^{'}(\tau_k) \right)^T \tilde{\boldsymbol{H}}^{'}(\tau_p) \boldsymbol{s}_p(n), \quad (51)$$

$$= \frac{-1}{\sigma_w^2} \sum_{n=1}^{N} \boldsymbol{s}_k(n) \left(\tilde{\boldsymbol{H}}^{'}(\tau_k) \right)^T \tilde{\boldsymbol{H}}^{'}(\tau_p) \boldsymbol{s}_p(n), \quad (52)$$

$$= \frac{-1}{\sigma_w^2} \mathrm{tr} \left[\mathcal{H}^{'}(\tau_k, \tau_p) \boldsymbol{R}_{kp}(n) \right], \quad (53)$$

where

$$\mathrm{tr} \left[\mathcal{H}^{'}(\tau_k, \tau_p) \boldsymbol{R}_{kp}(n) \right] \triangleq \sum_{n=1}^{N} \boldsymbol{s}_k(n) \left(\tilde{\boldsymbol{H}}^{'}(\tau_k) \right)^T \tilde{\boldsymbol{H}}^{'}(\tau_p) \boldsymbol{s}_p(n), \quad (54)$$

$$\mathcal{H}^{'}(\tau_k, \tau_p) = \left(\tilde{\boldsymbol{H}}^{'}(\tau_k) \right)^T \tilde{\boldsymbol{H}}^{'}(\tau_p), \quad (55)$$

and

$$\boldsymbol{R}_{kp}(n) = \sum_{n=1}^{N} \boldsymbol{s}_p(n) \boldsymbol{s}_k^T(n). \quad (56)$$

D. Derivation of $\frac{\partial^2 L(\boldsymbol{\theta})}{\partial s_k(n) \partial s_p(n)}$

1. If $k = p$

$$
\begin{aligned}
\frac{\partial L^2(\boldsymbol{\theta})}{\partial s_k(n)\partial s_k(n)} &= \frac{1}{\sigma_w^2}\frac{\partial}{\partial s_k(n)}\sum_{n=1}^{N}\tilde{\boldsymbol{H}}_0^T(\tau_k)\boldsymbol{u}(n), && (57)\\
&= \frac{1}{\sigma_w^2}\sum_{n=1}^{N}\tilde{\boldsymbol{H}}_0^T(\tau_k)\frac{\partial \boldsymbol{u}(n)}{\partial s_k(n)}, && (58)\\
&= \frac{-1}{\sigma_w^2}\sum_{n=1}^{N}\tilde{\boldsymbol{H}}_0^T(\tau_k)\tilde{\boldsymbol{H}}_0(\tau_k), && (59)\\
&= \frac{-N}{\sigma_w^2}\tilde{\boldsymbol{H}}_0^T(\tau_k)\tilde{\boldsymbol{H}}_0(\tau_k). && (60)
\end{aligned}
$$

2. If $k \neq p$

$$
\begin{aligned}
\frac{\partial L^2(\boldsymbol{\theta})}{\partial s_k(n)\partial s_p(n)} &= \frac{1}{\sigma_w^2}\frac{\partial}{\partial s_p(n)}\sum_{n=1}^{N}\tilde{\boldsymbol{H}}_0^T(\tau_k)\boldsymbol{u}(n), && (61)\\
&= \frac{1}{\sigma_w^2}\sum_{n=1}^{N}\tilde{\boldsymbol{H}}_0^T(\tau_k)\frac{\partial \boldsymbol{u}(n)}{\partial s_p(n)}, && (62)\\
&= \frac{-1}{\sigma_w^2}\sum_{n=1}^{N}\tilde{\boldsymbol{H}}_0^T(\tau_k)\tilde{\boldsymbol{H}}_0(\tau_p), && (63)\\
&= \frac{-N}{\sigma_w^2}\tilde{\boldsymbol{H}}_0^T(\tau_k)\tilde{\boldsymbol{H}}_0(\tau_p), && (64)\\
& && (65)
\end{aligned}
$$

E. Derivation of $\frac{\partial^2 L(\boldsymbol{\theta})}{\partial s_k(n)\partial \tau_p(n)}$

1. $k = p$

$$
\begin{aligned}
\frac{\partial L^2(\boldsymbol{\theta})}{\partial s_k(n)\partial \tau_k} &= \frac{1}{\sigma_w^2}\frac{\partial}{\partial \tau_k}\sum_{n=1}^{N}\tilde{\boldsymbol{H}}_0^T(\tau_k)\boldsymbol{u}(n), && (66)\\
&= \frac{1}{\sigma_w^2}\sum_{n=1}^{N}\left\{\left(\frac{\partial \tilde{\boldsymbol{H}}_0(\tau_k)}{\partial \tau_k}\right)^T\boldsymbol{u}(n) + \tilde{\boldsymbol{H}}_0^T(\tau_k)\frac{\partial \boldsymbol{u}(n)}{\partial \tau_k}\right\}, && (67)\\
&= \frac{1}{\sigma_w^2}\sum_{n=1}^{N}\left\{\left(\tilde{\boldsymbol{H}}_0^{'}(\tau_k)\right)^T\boldsymbol{u}(n) - \tilde{\boldsymbol{H}}_0^T(\tau_k)\tilde{\boldsymbol{H}}^{'}(\tau_k)\boldsymbol{s_k}(n)\right\}. && (68)
\end{aligned}
$$

2. $k \neq p$

$$\frac{\partial L^2(\boldsymbol{\theta})}{\partial s_k(n)\partial\tau_p} = \frac{1}{\sigma_w^2}\frac{\partial}{\partial\tau_p}\sum_{n=1}^{N}\tilde{\boldsymbol{H}}_0^T(\tau_k)\boldsymbol{u}(n), \tag{69}$$

$$= \frac{1}{\sigma_w^2}\sum_{n=1}^{N}\tilde{\boldsymbol{H}}_0^T(\tau_k)\frac{\partial\boldsymbol{u}(n)}{\partial\tau_p}, \tag{70}$$

$$= \frac{-1}{\sigma_w^2}\sum_{n=1}^{N}\tilde{\boldsymbol{H}}_0^T(\tau_k)\tilde{\boldsymbol{H}}^{'}(\tau_p)\boldsymbol{s_p}(n). \tag{71}$$

F. Fisher Information Matrix

Define the Fisher Information Matrix by $\boldsymbol{J} \in \mathcal{R}^{(KN+K)\times(KN+K)}$ as follows

$$\boldsymbol{J} = -\begin{bmatrix} E\left[\frac{\partial L^2(\boldsymbol{\theta})}{\partial\tau_0\partial\boldsymbol{\tau}}\right] & E\left[\frac{\partial L^2(\boldsymbol{\theta})}{\partial\tau_0\partial\boldsymbol{s}_0(n)}\right] & \cdots & E\left[\frac{\partial L^2(\boldsymbol{\theta})}{\partial\tau_0\partial\boldsymbol{s}_{K-1}(n)}\right] \\ \vdots & \vdots & & \vdots \\ E\left[\frac{\partial L^2(\boldsymbol{\theta})}{\partial\tau_{K-1}\partial\boldsymbol{\tau}}\right] & E\left[\frac{\partial L^2(\boldsymbol{\theta})}{\partial\tau_{K-1}\partial\boldsymbol{s}_0(n)}\right] & \cdots & E\left[\frac{\partial L^2(\boldsymbol{\theta})}{\partial\tau_{K-1}\partial\boldsymbol{s}_{K-1}(n)}\right] \\ E\left[\frac{\partial L^2(\boldsymbol{\theta})}{\partial\boldsymbol{s}_0(n)\partial\boldsymbol{\tau}}\right] & E\left[\frac{\partial L^2(\boldsymbol{\theta})}{\partial\boldsymbol{s}_0(n)\partial\boldsymbol{s}_0(n)}\right] & \cdots & E\left[\frac{\partial L^2(\boldsymbol{\theta})}{\partial\boldsymbol{s}_0(n)\partial\boldsymbol{s}_{K-1}(n)}\right] \\ \vdots & \vdots & & \vdots \\ E\left[\frac{\partial L^2(\boldsymbol{\theta})}{\partial\boldsymbol{s}_{K-1}(n)\partial\boldsymbol{\tau}}\right] & E\left[\frac{\partial L^2(\boldsymbol{\theta})}{\partial\boldsymbol{s}_{K-1}(n)\partial\boldsymbol{s}_0(n)}\right] & \cdots & E\left[\frac{\partial L^2(\boldsymbol{\theta})}{\partial\boldsymbol{s}_{K-1}(n)\partial\boldsymbol{s}_{K-1}(n)}\right] \end{bmatrix}, \tag{72}$$

where $E\left[\frac{\partial L^2(\boldsymbol{\theta})}{\partial\tau_k\partial\boldsymbol{\tau}}\right] \in \mathcal{R}^{1\times K}$ is defined as:

$$E\left[\frac{\partial L^2(\boldsymbol{\theta})}{\partial\tau_k\partial\boldsymbol{\tau}}\right] \triangleq \left[E\left[\frac{\partial L^2(\boldsymbol{\theta})}{\partial\tau_k\partial\tau_0}\right], E\left[\frac{\partial L^2(\boldsymbol{\theta})}{\partial\tau_k\partial\tau_1}\right], \ldots, E\left[\frac{\partial L^2(\boldsymbol{\theta})}{\partial\tau_k\partial\tau_{K-1}}\right]\right], \tag{73}$$

and $E\left[\frac{\partial L^2(\boldsymbol{\theta})}{\partial\tau_p\partial\boldsymbol{s}_k(n)}\right] \in \mathcal{R}^{1\times N}$ is defined as:

$$E\left[\frac{\partial L^2(\boldsymbol{\theta})}{\partial\tau_p\partial\boldsymbol{s}_k(n)}\right] \triangleq \left[E\left[\frac{\partial L^2(\boldsymbol{\theta})}{\partial\tau_p\partial s_k(1)}\right], E\left[\frac{\partial L^2(\boldsymbol{\theta})}{\partial\tau_p\partial s_k(2)}\right], \ldots, E\left[\frac{\partial L^2(\boldsymbol{\theta})}{\partial\tau_p\partial s_k(N)}\right]\right]. \tag{74}$$

Substituting equations (42), (53), (60), (65), (68), and (71), respectively, for $k, p = 0, 1, \ldots, K + KN - 1$ into the matrix in (72), and defining the inverse of the resulting matrix, we can obtain the Cramer-Roa Lower Bound of the estimates in $\boldsymbol{\theta}$.

G. Derivatives of the interpolation function

$$h_l(m\tau_k) \triangleq \frac{\sin\pi f_c(t_l - m\tau_k)}{\pi f_c(t_l - m\tau_k)}, \qquad l = 0, 1, ..., L-1, \tag{75}$$

where $t_l = lT_s$. The first derivative of $h_l(m\tau_k)$ with respect to τ_k is given as follows:

$$h_l^{'}(m\tau_k) \triangleq \frac{dh_l(m\tau_k)}{d\tau_k}, \tag{76}$$

$$= \frac{d}{d\tau_k}\left\{\frac{\sin\pi f_c(t_l - m\tau_k)}{\pi f_c(t_l - m\tau_k)}\right\}, \tag{77}$$

$$= \frac{-m\pi f_c(t_l - m\tau_k)\cos\pi f_c(t_l - m\tau_k) + m\sin\pi f_c(t_l - m\tau_k)}{\pi f_c(t_l - m\tau_k)^2}. \tag{78}$$

Accordingly, the second derivative of $h_l(m\tau_k)$ with respect to τ_k is given as follows:

$$h_l^{''}(m\tau_k) \triangleq \frac{d}{d\tau_k}h_l^{'}(m\tau_k), \tag{79}$$

$$= \frac{\left(2-(\pi f_c)^2(t_l-m\tau_k)^2\right)m^2\sin\pi f_c(t_l-m\tau_k)-2m^2\pi f_c(t_l-m\tau_k)\cos\pi f_c(t_l-m\tau_k)}{\pi f_c(t_l-m\tau_k)^4} \tag{80}$$