

Derivation of the importance sampling function for Wideband Array Signal Processing Using Sequential MC Methods

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I. DERIVATION OF THE IMPORTANCE SAMPLING FUNCTION

A. Derivation of the Gradient Vectors

The gradient vector

$$\nabla \mathcal{L}(\boldsymbol{\tau}_t) = \nabla \mathcal{L}_z(\boldsymbol{\tau}_t) + \nabla \mathcal{L}_\tau(\boldsymbol{\tau}_t), \quad (1)$$

where

$$\nabla \mathcal{L}_z(\boldsymbol{\tau}_t) = \frac{\partial}{\partial \boldsymbol{\tau}_t} \log \left(p(\mathbf{z}_t | \boldsymbol{\alpha}_t^{(i)}) \right), \quad (2)$$

$$\nabla \mathcal{L}_\tau(\boldsymbol{\tau}_t) = \frac{\partial}{\partial \boldsymbol{\tau}_t} \log \left(p(\boldsymbol{\tau}_t^{(i)} | \boldsymbol{\tau}_{t-1}^{(i)}, \mathbf{z}_t) \right). \quad (3)$$

We first present the derivation of $\nabla \mathcal{L}_z(\boldsymbol{\tau}_t)$ and then that of $\nabla \mathcal{L}_\tau(\boldsymbol{\tau}_t)$. Some details in the derivation can be found in [1].

A.1 Derivation of $\nabla \mathcal{L}_z(\boldsymbol{\tau}_t)$

$$\nabla \mathcal{L}_z(\boldsymbol{\tau}_t) = [\nabla \mathcal{L}_z(\tau_{t,0}), \nabla \mathcal{L}_z(\tau_{t,1}), \dots, \nabla \mathcal{L}_z(\tau_{t,k_t-1})]^T, \quad (4)$$

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where

$$\begin{aligned}
\nabla \mathcal{L}_z(\tau_{t,k}) &= \frac{\partial}{\partial \tau_{t,k}} \log(p(\mathbf{z}_t | \boldsymbol{\alpha}_t)), \\
&= \frac{\partial}{\partial \tau_{t,k}} \left\{ \frac{-1}{2\sigma_w^2} \left(\mathbf{z}_t - \tilde{\mathbf{H}}_0(\tau_t) \mathbf{a}_t \right)^T \left(\mathbf{z}_t - \tilde{\mathbf{H}}_0(\tau_t) \mathbf{a}_t \right) + \kappa_{\sigma_w^2} \right\}, \\
&= \frac{1}{\sigma_w^2} \sum_{l=0}^{L-1} \left(\mathbf{z}_t - \tilde{\mathbf{H}}_0(\tau_t) \mathbf{a}_t \right)^T \tilde{\mathbf{H}}'_l(\tau_{t,k}) \mathbf{s}_k(t-l), \\
&= \frac{1}{\sigma_w^2} \boldsymbol{\varepsilon}_t^T \tilde{\mathbf{H}}'(\tau_{t,k}) \mathbf{s}_k(t),
\end{aligned} \tag{5}$$

where $\boldsymbol{\varepsilon}_t$ is given by

$$\begin{aligned}
\boldsymbol{\varepsilon}_t &= \mathbf{z}_t - \tilde{\mathbf{H}}_0(\tau_t) \mathbf{a}_t, \\
&= \mathbf{y}_t - \sum_{l=0}^{L-1} \tilde{\mathbf{H}}_l(\tau_t) \mathbf{a}_{t-l},
\end{aligned}$$

where $\kappa_{\sigma_w^2}$ is a function of the noise variance σ_w^2 , and for $k = 0, \dots, k_t - 1$

$$\tilde{\mathbf{H}}'_l(\tau_{t,k}) \triangleq \frac{\partial \tilde{\mathbf{H}}_l(\tau_{t,k})}{\partial \tau_{t,k}}, \tag{6}$$

$$\begin{aligned}
\tilde{\mathbf{H}}'(\tau_{t,k}) &= \frac{\partial}{\partial \tau_{t,k}} \tilde{\mathbf{H}}(\tau_{t,k}), \\
&= \left[\tilde{\mathbf{H}}'_0(\tau_{t,k}), \tilde{\mathbf{H}}'_1(\tau_{t,k}), \dots, \tilde{\mathbf{H}}'_{L-1}(\tau_{t,k}) \right],
\end{aligned} \tag{7}$$

and $\mathbf{s}_k(t)$ is the signal amplitude for the k th source, defined as

$$\mathbf{s}_k(t) = [s_k(t), s_k(t-1), \dots, s_k(t-L+1)]^T. \tag{8}$$

A.2 Derivation of $\nabla \mathcal{L}_\tau(\boldsymbol{\tau}_t)$

$$\nabla \mathcal{L}_\tau(\boldsymbol{\tau}_t) = [\nabla \mathcal{L}_\tau(\tau_{t,0}), \nabla \mathcal{L}_\tau(\tau_{t,1}), \dots, \nabla \mathcal{L}_\tau(\tau_{t,k_t-1})]^T, \tag{9}$$

where

$$\begin{aligned}
\nabla \mathcal{L}_\tau(\tau_{t,k}) &= \frac{\partial}{\partial \tau_{t,k}} \log(p(\boldsymbol{\tau}_t | \boldsymbol{\tau}_{t-1})), \\
&= \frac{\partial}{\partial \tau_{t,k}} \left\{ \frac{-1}{2\sigma_v^2} (\boldsymbol{\tau}_t - \boldsymbol{\tau}_{t-1})^T (\boldsymbol{\tau}_t - \boldsymbol{\tau}_{t-1}) + \kappa_{\sigma_v^2} \right\}, \\
&= \frac{-1}{\sigma_v^2} (\tau_{t,k} - \tau_{t-1,k}),
\end{aligned} \tag{10}$$

where $\kappa_{\sigma_v^2}$ is a function of the noise variance σ_v^2 .

As a result, the k th element of the gradient vector $\nabla \mathcal{L}(\boldsymbol{\tau}_t)$ can be expressed as

$$\begin{aligned}
[\nabla \mathcal{L}(\boldsymbol{\tau}_t)]_k &= \nabla \mathcal{L}_z(\tau_{t,k}) + \nabla \mathcal{L}_\tau(\tau_{t,k}), \\
&= \frac{1}{\sigma_w^2} \boldsymbol{\varepsilon}_t^T \tilde{\mathbf{H}}'(\tau_{t,k}) \mathbf{s}_k(t) - \frac{1}{\sigma_v^2} (\tau_{t,k} - \tau_{t-1,k}).
\end{aligned} \tag{11}$$

B. Derivation of the Hessian Matrices

The gradient vector

$$\nabla^2 \mathcal{L}(\boldsymbol{\tau}_t) = \nabla^2 \mathcal{L}_z(\boldsymbol{\tau}_t) + \nabla^2 \mathcal{L}_\tau(\boldsymbol{\tau}_t), \quad (12)$$

where the k, p th elements of $\nabla^2 \mathcal{L}_z(\boldsymbol{\tau}_t)$ and $\nabla^2 \mathcal{L}_\tau(\boldsymbol{\tau}_t)$ are given by

$$[\nabla^2 \mathcal{L}_z(\boldsymbol{\tau}_t)]_{k,p} = \frac{\partial^2}{\partial \tau_{t,k} \partial \tau_{t,p}} \log \left(p(\mathbf{z}_t | \boldsymbol{\alpha}_t^{(i)}) \right), \quad k, p = 0, \dots, k_t - 1 \quad (13)$$

$$[\nabla^2 \mathcal{L}_\tau(\boldsymbol{\tau}_t)]_{k,p} = \frac{\partial^2}{\partial \tau_{t,k} \partial \tau_{t,p}} \log \left(p(\boldsymbol{\tau}_t^i | \boldsymbol{\tau}_{t-1}^{(i)}, \mathbf{z}_t) \right), \quad k, p = 0, \dots, k_t - 1. \quad (14)$$

B.1 Derivation of $[\nabla^2 \mathcal{L}_z(\boldsymbol{\tau}_t)]_{k,p}$

1. If $k \neq p$

$$\begin{aligned} [\nabla^2 \mathcal{L}_z(\boldsymbol{\tau}_t)]_{k,p} &= \frac{\partial}{\partial \tau_{t,p}} \left\{ \frac{\partial}{\partial \tau_{t,k}} \log \left(p(\mathbf{z}_t | \boldsymbol{\alpha}_t^{(i)}) \right) \right\}, \\ &= \frac{\partial}{\partial \tau_{t,p}} \nabla \mathcal{L}_z(\tau_{t,k}), \\ &= \frac{\partial}{\partial \tau_{t,p}} \left\{ \frac{1}{\sigma_w^2} \boldsymbol{\varepsilon}_t^T \tilde{\mathbf{H}}'(\tau_{t,k}) \mathbf{s}_k(t) \right\}, \\ &= \frac{-1}{\sigma_w^2} \mathbf{s}_p^T(t) \left(\tilde{\mathbf{H}}'(\tau_{t,p}) \right)^T \tilde{\mathbf{H}}'(\tau_{t,k}) \mathbf{s}_k(t), \end{aligned} \quad (15)$$

2. If $k = p$

$$\begin{aligned} [\nabla^2 \mathcal{L}_z(\boldsymbol{\tau}_t)]_{k,k} &= \frac{\partial^2}{\partial \tau_{t,k}^2} \log \left(p(\mathbf{z}_t | \boldsymbol{\alpha}_t^{(i)}) \right), \\ &= \frac{\partial}{\partial \tau_{t,k}} \nabla \mathcal{L}_z(\tau_{t,k}), \\ &= \frac{\partial}{\partial \tau_{t,k}} \left\{ \frac{1}{\sigma_w^2} \boldsymbol{\varepsilon}_t^T \tilde{\mathbf{H}}'(\tau_{t,k}) \mathbf{s}_k(t) \right\}, \\ &= \frac{1}{\sigma_w^2} \left\{ \boldsymbol{\varepsilon}_t^T \tilde{\mathbf{H}}''(\tau_{t,k}) \mathbf{s}_k(t) - \mathbf{s}_p^T(t) \left(\tilde{\mathbf{H}}'(\tau_{t,p}) \right)^T \tilde{\mathbf{H}}'(\tau_{t,k}) \mathbf{s}_k(t) \right\}, \end{aligned} \quad (16)$$

where

$$\tilde{\mathbf{H}}''(\tau_{t,k}) = \frac{\partial}{\partial \tau_{t,k}} \tilde{\mathbf{H}}'(\tau_{t,k}). \quad (17)$$

B.2 Derivation of $[\nabla^2 \mathcal{L}_\tau(\boldsymbol{\tau}_t)]_{k,p}$

1. If $k \neq p$

$$\begin{aligned}
[\nabla^2 \mathcal{L}_\tau(\boldsymbol{\tau}_t)]_{k,p} &= \frac{\partial}{\partial \tau_{t,p}} \left\{ \frac{\partial}{\partial \tau_{t,k}} \log(p(\boldsymbol{\tau}_t | \boldsymbol{\tau}_{t-1})) \right\}, \\
&= \frac{\partial}{\partial \tau_{t,p}} \nabla \mathcal{L}_\tau(\tau_{t,k}), \\
&= \frac{\partial}{\partial \tau_{t,p}} \left\{ \frac{1}{\sigma_v^2} (\tau_{t,k} - \tau_{t-1,k}) \right\} \\
&= 0.
\end{aligned} \tag{18}$$

2. If $k = p$

$$[\nabla^2 \mathcal{L}_\tau(\boldsymbol{\tau}_t)]_{k,k} = \frac{\partial^2}{\partial \tau_{t,k}^2} \log(p(\boldsymbol{\tau}_t | \boldsymbol{\tau}_{t-1})), \tag{19}$$

$$= \frac{\partial}{\partial \tau_{t,k}} \nabla \mathcal{L}_\tau(\tau_{t,k}), \tag{20}$$

$$= \frac{\partial}{\partial \tau_{t,k}} \left\{ \frac{1}{\sigma_v^2} (\tau_{t,k} - \tau_{t-1,k}) \right\}, \tag{21}$$

$$= \frac{1}{\sigma_v^2}. \tag{22}$$

Therefore, the k, p th element of the Hessian matrix $\nabla^2 \mathcal{L}(\boldsymbol{\tau}_t)$ can be expressed as follows

$$[\nabla^2 \mathcal{L}(\boldsymbol{\tau}_t)]_{k,p} = [\nabla^2 \mathcal{L}_z(\boldsymbol{\tau}_t)]_{k,p} + [\nabla^2 \mathcal{L}_\tau(\boldsymbol{\tau}_t)]_{k,p}, \tag{23}$$

$$= \begin{cases} -\sigma_w^{-2} \mathbf{s}_k^T(t) \left(\tilde{\mathbf{H}}'(\tau_{t,k}) \right)^T \tilde{\mathbf{H}}'(\tau_{t,k}) \mathbf{s}_k(t), & \text{if } k = p, \\ \sigma_w^{-2} \left\{ \boldsymbol{\varepsilon}_t^T \tilde{\mathbf{H}}''(\tau_{t,k}) \mathbf{s}_k(t) - \mathbf{s}_p^T(t) \left(\tilde{\mathbf{H}}'(\tau_{t,p}) \right)^T \tilde{\mathbf{H}}'(\tau_{t,k}) \mathbf{s}_k(t) \right\} + \sigma_v^{-2}, & \text{if } k \neq p. \end{cases} \tag{24}$$

REFERENCES

- [1] W. Ng, J. P. Reilly, and T. Kirubarajan, "The derivation of the theoretical CRLB for wide-band array signal processing: Static scenario," May 2002. Please download the document at <http://www.ece.mcmaster.ca/~reilly>.