Multimedia Communications

Mathematical Preliminaries for Lossy compression

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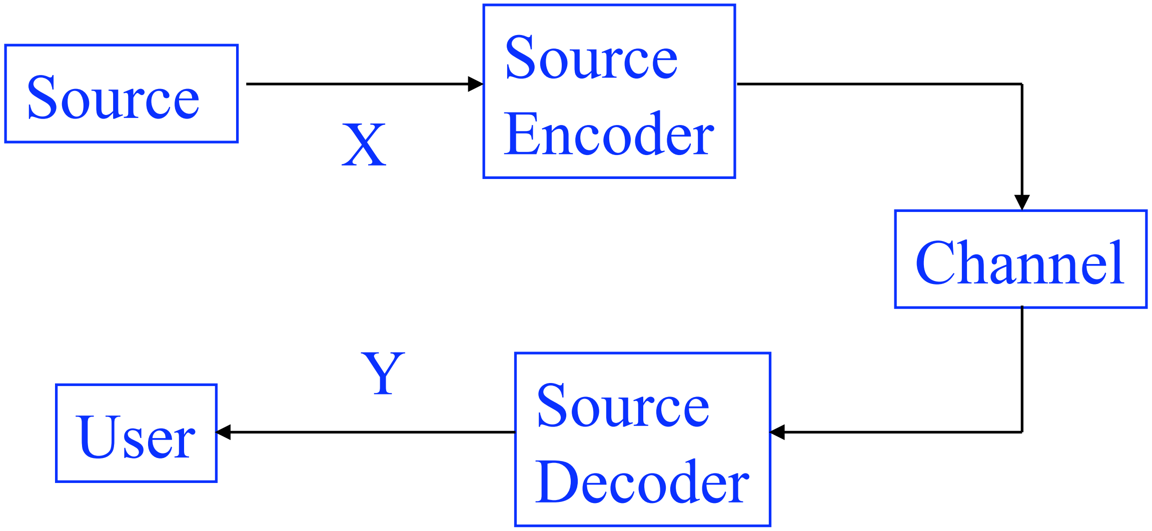
Lossy compression

- In a lossless compression, the reconstructed signal is identical to the original sequence
- Only a limited amount of compression can be obtained with lossless compression
- There is a floor (entropy of the source) below which we cannot drive the size of the compressed sequence
- In some applications consequences of loss of information may prohibit us from loss of information (bank records, medical images)

Lossy compression

- If the resources are limited and we do not require absolute integrity, we can improve the amount of compression by accepting certain degree of loss during compression
- Lossless compression: rate is the only performance measure
- In lossy compression rate by itself is not enough
- A measure of difference between original and reconstructed data
- Our goal: incur minimum amount of distortion while compressing to the lowest possible rate
- There is a tradeoff between minimizing rate and keeping distortion small

Lossy compression



- X can be modeled as a random variable
- Y is also a random variable

Lossy compression

- Exp: suppose source output consists of {0,1,2,…,15}. Source encoder quantizes the output into one of {0,1,2,…,7}. Source decoder inverse-quantizes the output of source encoder to {0,2,..,14}
- X={0,1,2,…, 15}: 16 elements, Y={0,2,4,…,14}: 8 elements
- Y is a random variable:
  - P(Y=0)=P(Y=0|X=0)P(X=0)+P(Y=0|X=1)P(X=1)
- Source encoder-decoder can be described by P(Y|X)

$$Q = \begin{bmatrix} p(y_1 \mid x_1) & p(y_1 \mid x_2) & & p(y_1 \mid x_N) \\ p(y_2 \mid x_1) & p(y_2 \mid x_2) & & \\ & & & \\ p(y_M \mid x_1) & p(y_M \mid x_2) & & p(y_M \mid x_N) \end{bmatrix}$$

$$\sum_{i=1}^{M} p(y_i \mid x_j) = 1$$

Joint Entropy

- Let X and Y be two information sources with alphabets A_N and B_M and joint probability distribution $P : A_N \times B_M \rightarrow [0,1]$.
- The self-information of the event (X=x,Y=y) is $i(x,y)=-\log P(x,y)$. The joint entropy of the two sources is

$$H(X,Y) = E[i(x,y)] = -\sum_{x \in A_N, y \in B_M} P(x,y) \log P(x,y)$$

- H(X,Y) is the average amount of information carried by a pair of values of the two information sources.
- $H(X,Y) \leq H(X) + H(Y)$ and the equality holds when the two sources are independent.

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Conditional Entropy

- The conditional self-information of the event (X=x|Y=y) is i(x|y)=-logP(x|y). The conditional entropy of the two sources is

$$H(X \mid Y) = E[i(x \mid y)] = -\sum_{x,y} P(x,y) \log P(x \mid y) = -\sum_{x,y} P(x,y) \log \frac{P(x,y)}{P(y)}$$

- H(X|Y) is the average amount of information carried by X when the value of Y is known.
- H(X|Y) + H(Y) = H(Y|X) + H(X) = H(X,Y)
- H(X|Y) ≤ H(X), equality holds for independent sources.

Mutual Information

- Probability of source symbols p(x): a priori probability
- After reconstructing y the probability of input symbol being x becomes p(x|y): a posteriori probability
- Difference between information (uncertainty) before and after reconstruction of y measures the gain in information due to y:

  i(x;y)= i(x)-i(x|y)

  =-log(p(x))+log(p(x|y))=log(p(x|y)/p(x))=log(p(x,y)/p(x)p(y))
- i(x,y) is called mutual information of the x and y.
- Average mutual information of the two sources is the information that Y carries about X:

$$I(X;Y) = E[i(x;y)] = -\sum_{x,y} P(x,y)\log\frac{P(x,y)}{P(x)P(y)}$$

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Mutual Information

- $I(X;Y) = I(Y;X)$
- $I(X;Y) \geq 0$, equality holds when the sources are independent.
- $I(X;Y) \leq H(X)$ and $I(X;Y) \leq H(Y)$
- $I(X;Y) = H(Y)-H(Y|X) = H(X)-H(X|Y) = H(X)+H(Y)-H(X,Y)$

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Distortion

- Distortion criteria:

1. Subjective criteria: measured by the effect the distortion has on the receiver. Not possible to model analytically.
2. Objective mathematical measures: use mathematical formula to measure distortion. Analytical or numerical optimizations are possible.
3. Objective measures in conjunction with models of the receiver (human auditory system, human visual system).
   – The process of human perception is very difficult to model and the models obtained are very complex

Mathematical Distortion Measures

- Average distortion

$$d_l(\underline{x}, \underline{y}) = \frac{1}{N}\sum_{i=1}^{N} |x - y|^l$$

- $l = 1 \rightarrow$ mean absolute error (MAE) $\quad d_l(x, y) = |x - y|$
- $l = 2 \rightarrow$ mean square error (MSE)
- $l = \infty \rightarrow$ maximum error $\quad d_\infty(\underline{x}, \underline{y}) = \max_i |x_i - y_i|$
- Sometimes the MSE is measured in dB:

$$\mathrm{SNR(dB)} = 10\log_{10}\frac{\sigma_x^2}{\sigma_d^2} \quad \mathrm{PSNR(dB)} = 10\log_{10}\frac{x_{max}^2}{\sigma_d^2}$$

- In a probabilistic modeling framework:

$$D(X,Y) = \sum_{x,y} P(x,y)d(x,y) = \sum_{x,y} P(x)P(y|x)d(x,y)$$

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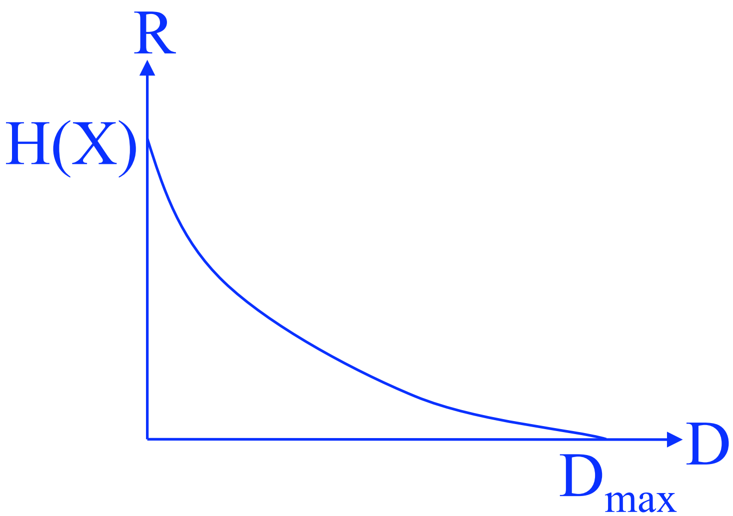
Rate-Distortion Function: Definition

- Main trade-off: number of bits vs. signal distortion
- R(D) specifies the lowest rate R at which the output of a source can be encoded while keeping the distortion ≤ D.



  - continuous
  - monotonic decreasing
  - convex
  - $R(0) = H(X)$
  - $R(D) = 0$ for $D \geq D_{max}$

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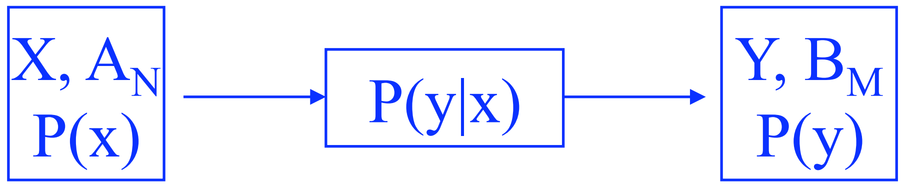
Rate-Distortion Function: Definition

- A coding scheme can be viewed as



  where Y is the reconstructed signal, A_N and P(x) describe the source and the conditional pdf P(y|x) and B_M describe the coding scheme.
- We can express the rate-distortion function of a source in terms of these functions.

Rate-Distortion Function: Definition

- The average distortion of a coding scheme is:

$$D(X,Y) = \sum_{x,y} P(x,y)d(x,y) = \sum_{x,y} P(x)P(y\,|\,x)d(x,y)$$

- To find R(D), we choose from the set of coding schemes that have a distortion smaller than D:

$$\Gamma = \left\{ P(y\,|\,x);\ D(X,Y) \leq D \right\}$$

the one that achieves the smallest rate R.

- R(D) is defined as

$$\mathrm{R(D)} = \min_{\mathrm{p(y|x)} \in \Gamma} \mathrm{I(X,Y)} = -\sum_{\mathrm{x,y}} \mathrm{P(x,y)} \log \frac{\mathrm{P(x\,|\,y)}}{\mathrm{P(x)}}$$

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Rate-Distortion Function: Computation

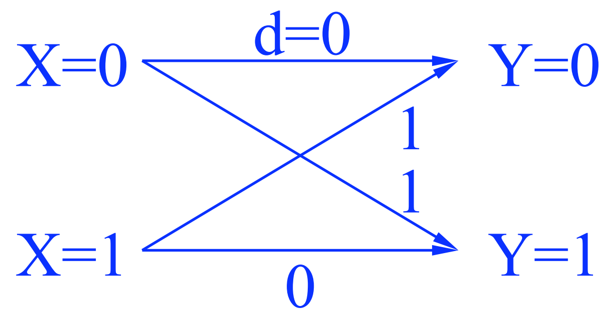
- How do we compute R(D)?
  - Analytically: find lower bound, show that it can be achieved.
  - Method of Lagrangian multipliers.
  - Computational approach: convex programming.

- In general, a difficult task.

- Can be found analytically only in a few simple cases.

Rate-Distortion Function: Example

- Consider a binary source with $P(0) = p$, $P(1) = 1-p$, $p<1/2$.
- Let the distortion function be $d(x,y) = x \oplus y$ (xor)



- Find the rate distortion function for this source.
- We follow two approaches: Lagrangian multipliers and an analytical method

Rate-Distortion Function: Example

- Source encoder-decoder:

$$Q = \begin{bmatrix} p(y0\,|\,x0) & p(y0\,|\,x1) \\ p(y1\,|\,x0) & p(y1\,|\,x1) \end{bmatrix} = \begin{bmatrix} a & 1-b \\ 1-a & b \end{bmatrix}$$

$$I(X;Y) = -\sum_{x,y} P(y\,|\,x)P(x)\log\frac{P(y\,|\,x)P(x)}{P(x)P(y)}$$

$$D(X,Y) = \sum_{x,y} P(x)P(y\,|\,x)d(x,y)$$

$$R(D) = \min I(x,y)$$

$$D(X,Y) < D$$

$$J(a,b,\lambda) = I(x,y) - \lambda(D(X,Y) - D)$$

$$R(D) = -p\log p - (1-p)\log(1-p) + D\log D + (1-D)\log(1-D)$$

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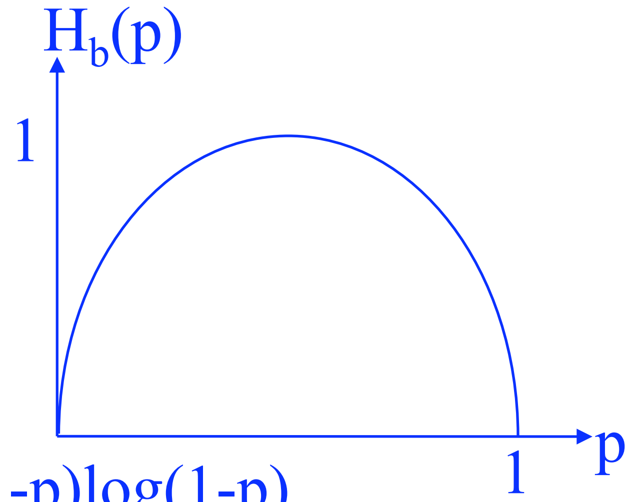
Rate-Distortion Function: Example



- $P(x=0) = p$, $P(x=1) = 1-p$, $p<1/2$.
- $d(x,y) = x \oplus y$ (xor)
- $R(0) = H(X) = -p\log p-(1-p)\log(1-p)$
- $H_b(p)$ is defined as $H_b(p) =-p\log p-(1-p)\log(1-p)$
- $H_b(p)= H_b(1-p)$
- $D(X,Y) = 1\cdot P(x=0,y = 1) + 1 \cdot P(x=1,y = 0) = P(x \oplus y = 1)$
- D_{max} is obtained when we send no bits, so we always decode Y=1 (D = p) or Y=0 (D = 1-p). The best of these two is Y=1, so $D_{max} = p$.

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Rate-Distortion Function: Example

- Then $\Gamma = \{P(y \mid x);\ D(X,Y) \le D\} = \{P(y \mid x);\ \mathrm{P}(\mathrm{x} \oplus \mathrm{y} = 1) \le D\}$
- The mutual information is $\mathrm{I(X,Y) = H(X) - H(X \mid Y)}$
- Since, when Y is known, we can obtain X ⊕ Y from X and vice versa, H(X|Y) = H(X ⊕ Y|Y) and then

$$\mathrm{I(X,Y) = H(X) - H(X \oplus Y \mid Y).}$$

- A lower bound on I(X,Y) is then

$$\begin{aligned} I(X,Y) &= H(X) - H(X \mid Y) \\ &= H(X) - H(X \oplus Y \mid Y) \\ &\ge H(X) - H(X \oplus Y) \\ &= H_b(p) - H_b(P(x \oplus y) = 1). \end{aligned}$$

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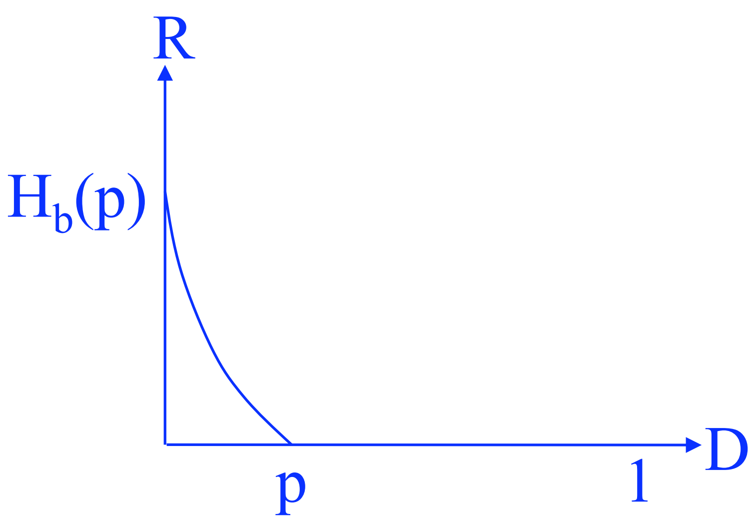
Rate-Distortion Function: Example

- But $P(x \oplus y = 1) \leq D$, so $I(X,Y) \geq H_b(p) - H_b(D)$.
- Let us now show that the lower bound is actually achievable: Let $P(X=0|Y=1) = P(X=1|Y=0) = D$. Then
  - $P(X \oplus Y = 1) = P(X=1,Y=0) + P(X=0,Y=1)$
    $= P(Y=0)P(X=1|Y=0) + P(Y=1)P(X=0|Y=1)$
    $= P(Y=0) \cdot D + P(Y=1) \cdot D = D$
  - $P(X \oplus Y = 0) = 1-D$
  - $I(X,Y) = H(X) - H(X \oplus Y) = H_b(p) - H_b(D)$, q.e.d.

Rate-Distortion Function: Example

- Putting it together,

$$R(D) = \begin{cases} H_b(p) - H_b(D), & D < p \\ 0, & \text{otherwise} \end{cases}$$



Continuous Amplitude Sources

- Continuous-amplitude memoryless sources are modeled as continuous RVs described by their pdf.
- Continuous RVs have infinite absolute entropy. We define the differential entropy as

$$h(X) = E\left[-\log p_X(x)\right] = -\int_{-\infty}^{+\infty} p_X(x)\log p_X(x)dx$$

- The entropy of a Gaussian random variable h(X) = 1/2 $\log(2\pi e\sigma^2)$
- For any source with variance σ^2, $h(X) \le h_G$.

Continuous Amplitude Sources

- The rate-distortion function of a continuous source is defined similarly with the discrete source:

$$I(X,Y) = -\int\int p(x,y)\log\frac{p(x\mid y)}{p(x)}dxdy$$

$$d(X,Y) = \int\int p(x)p(y\mid x)d(x,y)dxdy$$

$$R(D) = \inf_{p(y|x)\in\Gamma} I(X,Y)$$

$$\Gamma = \{p(y\mid x);\ d(X,Y)\le D\}$$

Rate distortion

- The rate-distortion function for a zero mean Gaussian with distortion function $d(x,y) = (x-y)^2$ is

$$R(D) = \begin{cases} \frac{1}{2}\log\frac{\sigma^2}{D}, & D < \sigma^2 \\ 0, & \text{otherwise} \end{cases}$$

- Rate-distortion function for the Gaussian source is larger than the rate distortion function for any other source with a continuous distribution and the same variance
- The following are the lower bounds for a random variable X:
- squared error $$R(D) = h(X) - \frac{1}{2}\log(2eD)$$
- Absolute value error $$R(D) = h(X) - \log(2eD)$$

Noisy Source Coding Theorem

- Let R(D) be the rate-distortion function of a stationary source. Then, for any $D>0$ and $\varepsilon>0$:
  - POSITIVE THEOREM: a source code of block length n (sufficiently large) exists that encodes source vectors $\underline{X}=\{X_1,\ldots, X_n\}$ at a rate $R < R(D) + \varepsilon$ with $d(X,Y) \leq D$.
  - NEGATIVE THEOREM: a source code does not exist that encodes the source at a rate $R < R(D)$ with $d(X,Y) \leq D$.

Models for Continuous Sources

- **Uniform distribution**: $$p_X(x) = \begin{cases} \frac{1}{b-a}, & x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$
  - The entropy $h(X) = \log(b-a)$.
- **Gaussian distribution**: $$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$
  - The entropy $h(X) = 1/2 \log(2\pi e\sigma^2) = h_G$
  - For any source with variance σ^2, $h(X) \leq h_G$.

Models for Continuous Sources

- **Exponential distribution**:
  - single-sided distribution

$$p_X(x) = \lambda \exp[-\lambda x],\ x \ge 0$$

- **Laplacian distribution**:
  - good for modeling peaked distributions

$$p_X(x) = \frac{1}{\sqrt{2\sigma^2}} \exp\left[-\frac{\sqrt{2}|x|}{\sigma}\right]$$

- **Gamma distribution**:
  - even more peaked

$$p_X(x) = \frac{3^{1/4}}{\sqrt{8\pi\sigma|x|}} \exp\left[-\frac{\sqrt{3}|x|}{2\sigma}\right]$$

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Models for Continuous Sources

- **Generalized Gaussian distribution**:
  - $c = 2$ Gaussian
  - $c = 1$ Laplacian
  - $c = 0.7$ good model for subbands

$$p_X(x) = a \exp[-b(x-\mu)^c]$$

$$a = \frac{bc}{2\Gamma(1/c)}$$

$$b = \frac{1}{\sigma}\sqrt{\frac{\Gamma(3/c)}{\Gamma(1/c)}}$$

Models for Sources with Memory

- Linear system models:

$$x_n = \sum_{i=1}^{N} a_i x_{n-i} + \sum_{j=1}^{M} b_j \varepsilon_{n-j} + \varepsilon_n,$$

where ε_n is a Gaussian white noise sequence with variance σ_ε^2.

  - LTI system with N poles and M zeros.
  - Auto-regressive moving average (ARMA) model.
  - For M = 0 we obtain the "all-pole" or "AR" or "Markov" model. Good model for speech production.

Models for Sources with Memory

- AR(N) is an N-th order Markov model

$$x_n = \sum_{i=1}^{N} a_i x_{n-i} + \varepsilon_n$$

- For the first-order Markov source with an autocorrelation function of $\phi_k = |r|^k$ the rate-distortion function is

$$R(D) = \begin{cases} \frac{1}{2}\log\frac{1-r^2}{D}, & D \le \frac{1-r}{1+r} \\ \text{(complex)}, & \text{otherwise} \end{cases}$$