Multimedia Communications

Scalar Quantization

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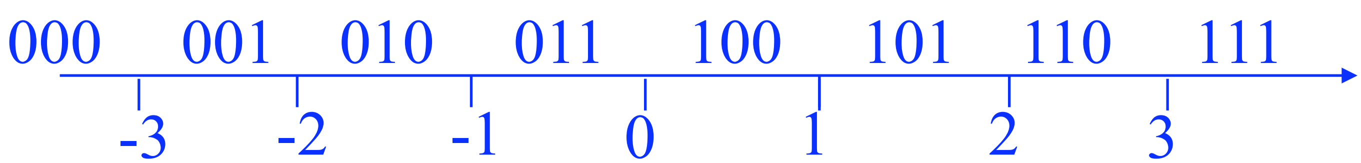
Scalar Quantization

- In many lossy compression applications we want to represent source outputs using a small number of code words.
- Process of representing a large set of values with a much smaller set is called quantization
- Quantizer consists of two mappings: encoder mapping and decoder mapping
- Encoder divides the range of values that the source generates into a number of intervals
- Each interval is represented by a distinct codeword



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Scalar Quantization

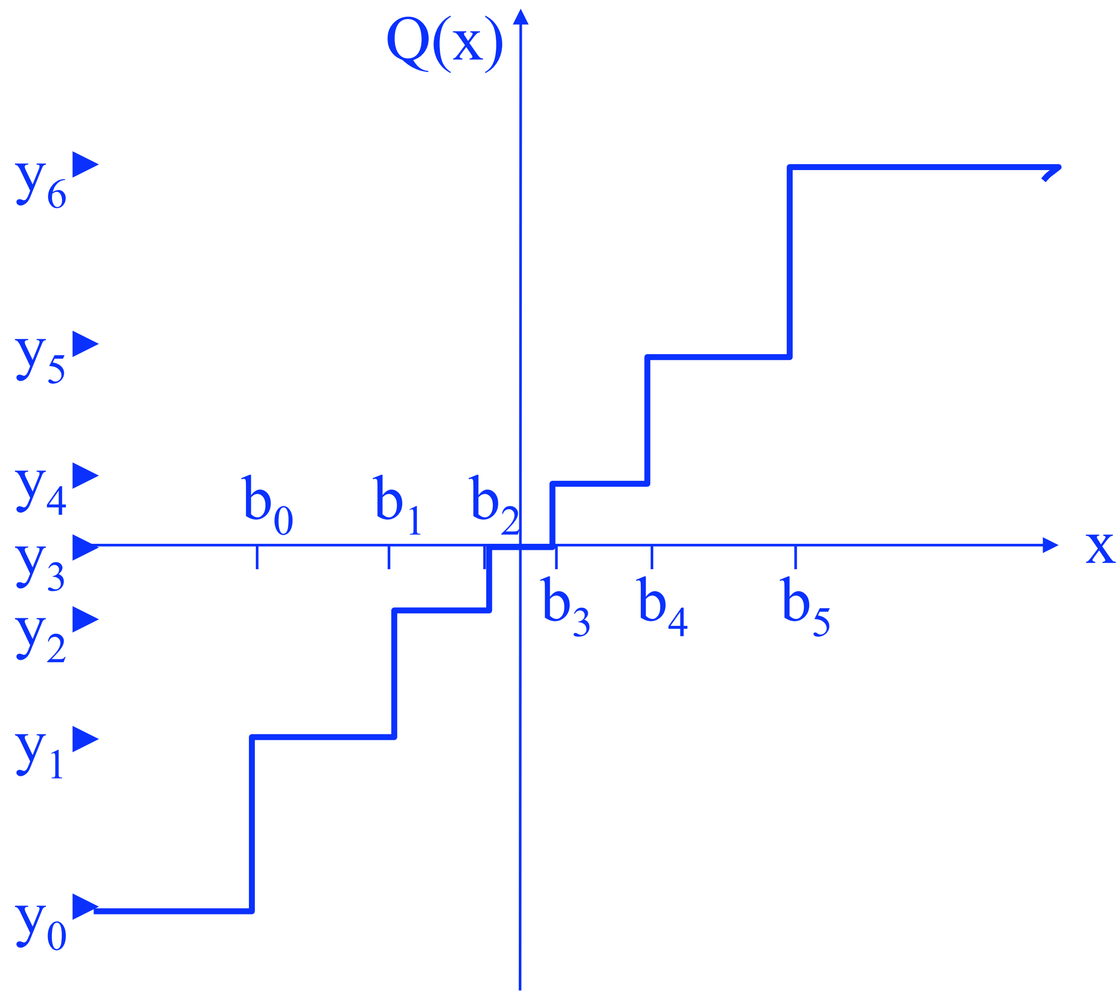
- All the source outputs that fall into a particular interval are represented by the codeword of that interval
- For every codeword generated by the encoder, the decoder generates a reconstruction value
- Because a codeword represents an entire interval, there is no way of knowing which value in the interval actually was generated by the source

Code	Output
000	-3.5
001	-2.5
010	-1.5
011	-0.5
100	0.5
101	1.5
110	2.5
111	3.5

Scalar Quantization

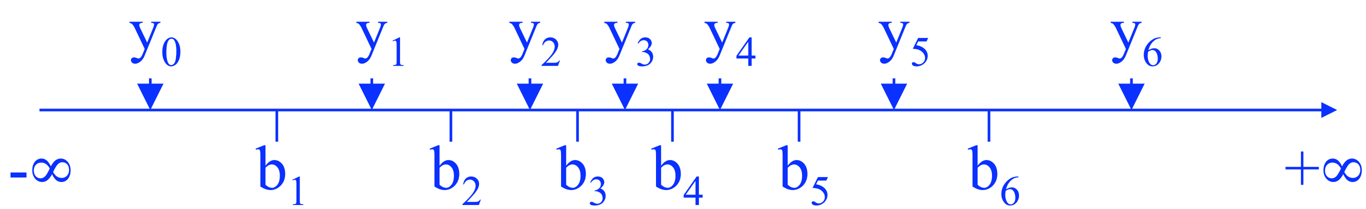
- Construction of intervals can be viewed as part of the design of the encoder
- Selection of reconstruction values is part of the design of the decoder
- The quality of the reconstruction depend on both intervals and reconstruction values
- The design is considered as a pair
- Design a quantizer: divide the input range into intervals, assign binary codes to these intervals, find reconstruction values
- Do all these while satisfying the rate-distortion criteria

Scalar Quantizers



Scalar Quantization

- Source: random variable X with pdf of $f_x(x)$
- M: number of intervals
- b_i, i=0, 1,2, .., M: M+1 end points of the intervals (decision boundary)
  - b_0 and b_M could be infinite
- y_i: M reconstruction level



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Scalar Quantization

- Distortion: mean squared quantization error

$$\delta_q^2 = E[(X - Q(X))^2] = \int_{-\infty}^{+\infty} (x - Q(x))^2 f_X(x)dx$$

$$\delta_q^2 = \sum_{i=1}^{M} \int_{b_{i-1}}^{b_i} (x - y_i)^2 f_X(x)dx$$

- Rate: if fixed-length codewords are used to represent the quantizer output, the rate is given by: $R = \lceil \log_2 M \rceil$
- Selection of decision boundaries will not affect the rate

Scalar Quantization

- If variable length codewords are used
  the rate will be:

$$R = \sum_{i=1}^{M} l_i P(y_i)$$

$$P(y_i) = \int_{b_{i-1}}^{b_i} f_x(x)dx$$

$$R = \sum_{i=1}^{M} l_i \int_{b_{i-1}}^{b_i} f_x(x)dx$$

- Selection of decision boundaries will
  affect the rate

	Code1	Code2
y1	1110	000
y2	1100	001
y3	100	010
y4	00	011
y5	01	100
y6	101	101
y7	1101	110
y8	1111	111

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Scalar Quantization

- Problem of finding optimum scalar quantizer:

1. given a distortion constraint $\sigma_q^2 \leq D^*$ find the decision boundaries, reconstruction levels, and binary codes that minimize the rate or
2. given a rate constraint R<R* find the decision boundaries, reconstruction levels, and binary codes that minimize the distortion

Scalar Quantizer

Fixed length codewords

Variable length codewords

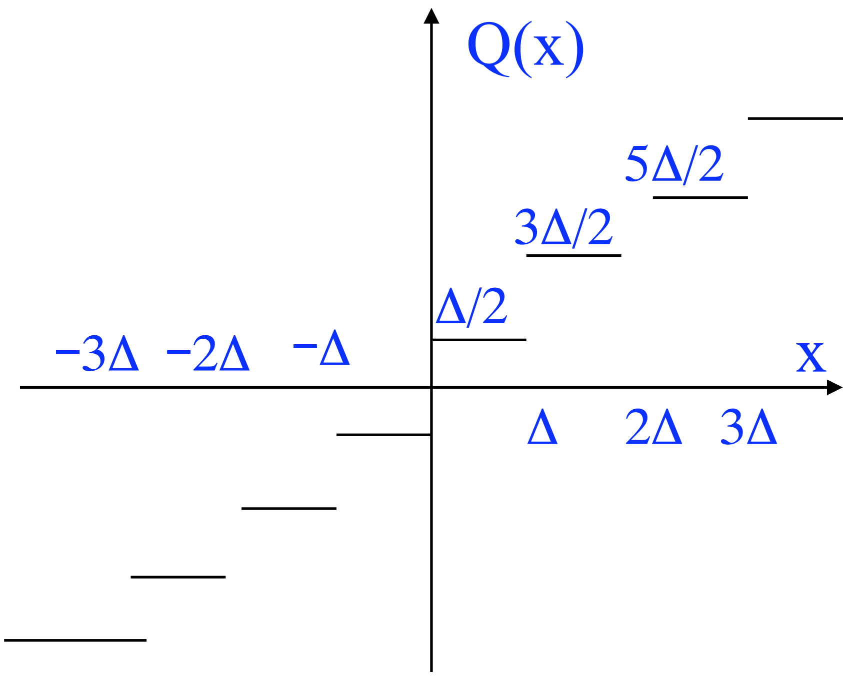
Uniform

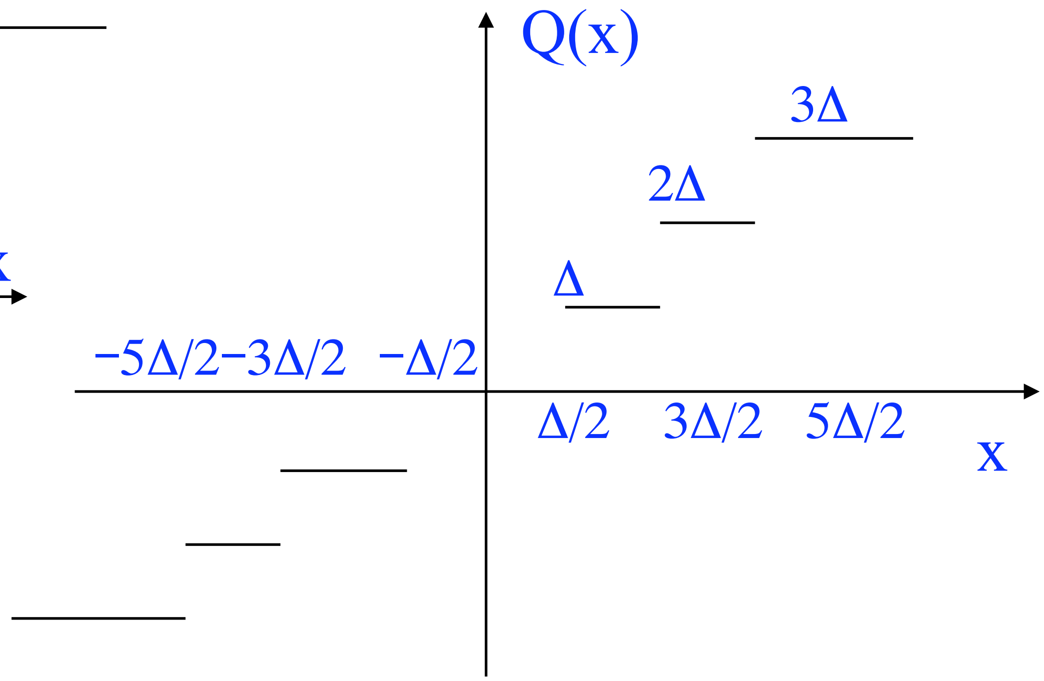
Non-uniform

Uniform Quantizers

- All intervals are the same size except possibly for the two outer intervals (decision boundaries are spaced evenly)
- Reconstruction values are also spaced evenly with the same spacing as the decision boundaries
- In the inner intervals the reconstruction values are the midpoint of the intervals
- If zero is not a reconstruction level of the quantizer, it is called a midrise quantizer (M is even)
- If zero is a reconstruction level of the quantizer, it is called a midtread quantizer (M is odd)

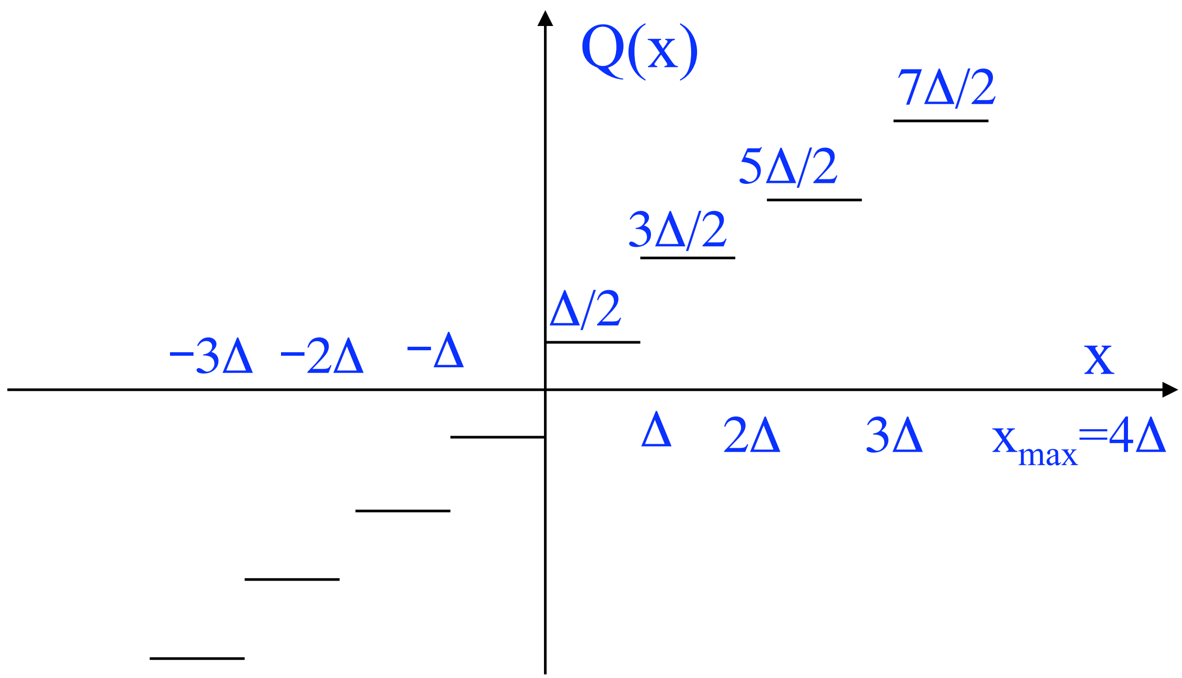
Uniform Quantizers





Uniform Quantizers for Uniform Source

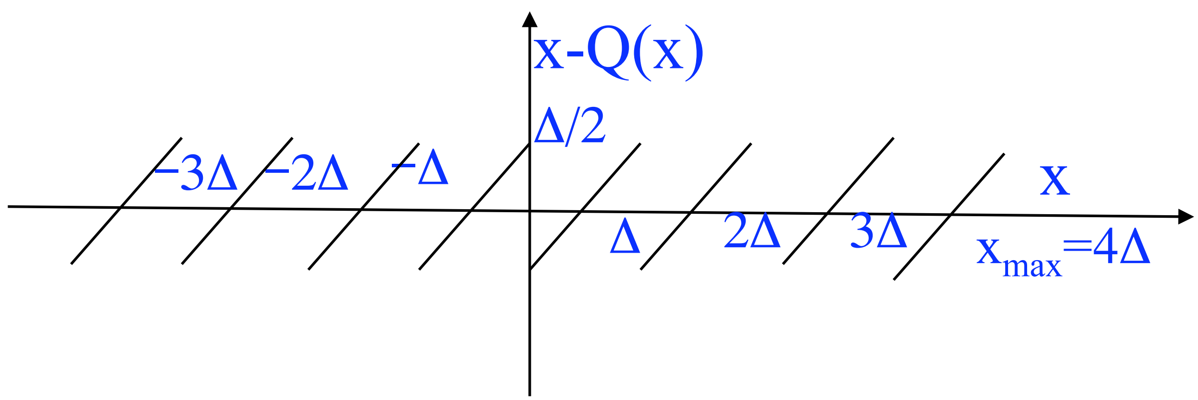
- Input: uniformly distributed in $[-x_{max}, x_{max}]$
- For an M-level uniform quantizer we divide $[-x_{max}, x_{max}]$ into M equally sized intervals each with a step size of $\Delta = \frac{2x_{max}}{M}$



Uniform Quantizers for Uniform Source

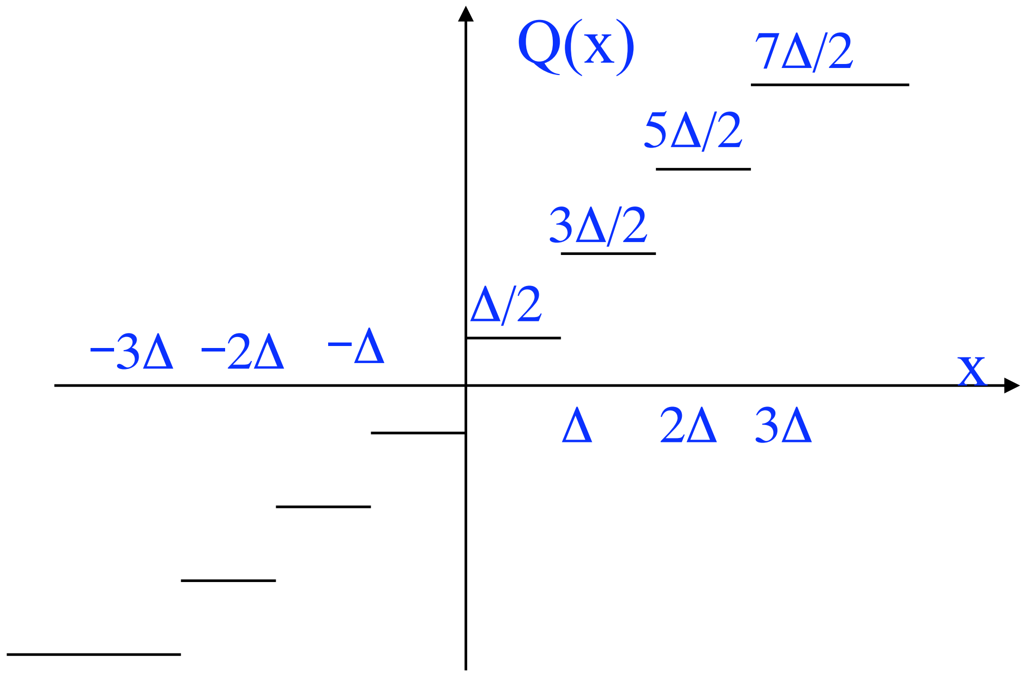
- For uniform distributions, quantization error q = Q(x) - x of a uniform quantizer is uniformly distributed in [-Δ/2,+ Δ /2]
- Then E[q] = 0, σ_q^2=E[q^2] = Δ^2/12.
- If the quantizer output is encoded using n bits per sample:

$$SNR(dB) = 10\log_{10}\frac{\sigma_x^2}{\sigma_q^2} = 10\log_{10} M^2 = 6.02\log_2 M = 6.02n$$



Uniform Quantizers for Non-uniform Source

- When the distribution is not uniform, it is not a good idea to find the step size by dividing the range of inputs by the number of levels (sometimes the range is infinite)
- If pdf is not uniform, an optimal quantization step size can be found that minimizes D.



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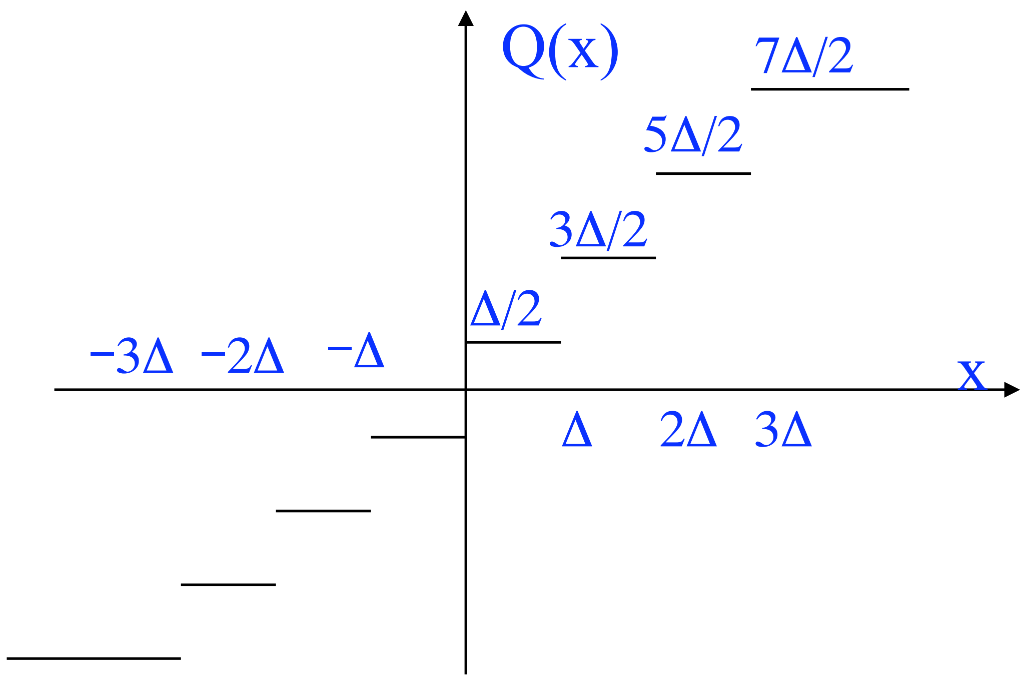
Uniform Quantizers for Non-uniform Source

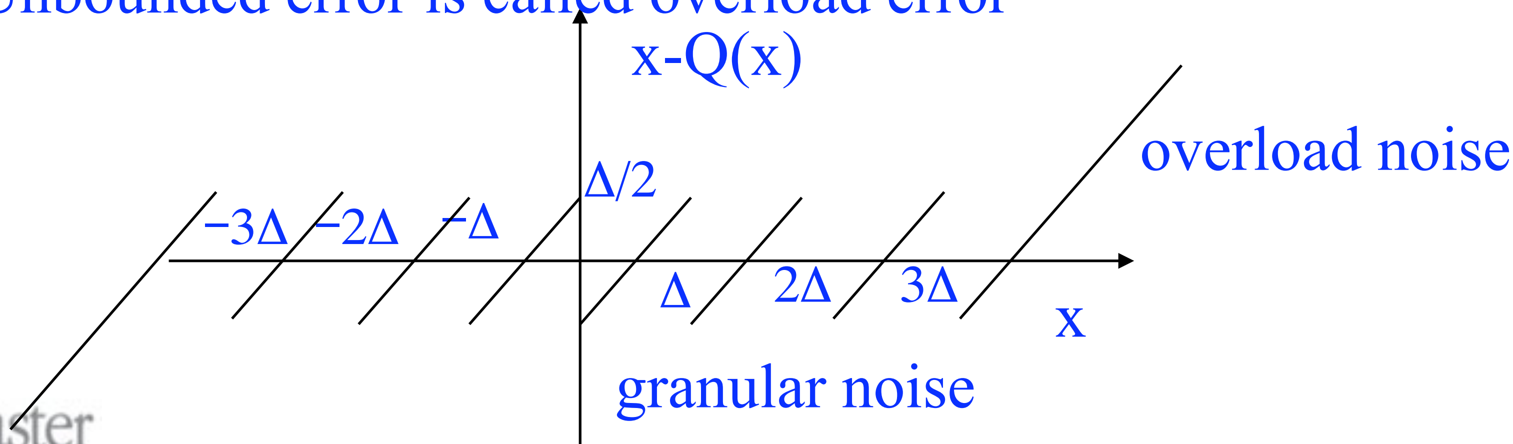
$$\sigma_q^2 = 2\sum_{i=1}^{M/2-1} \int_{(i-1)\Delta}^{i\Delta} \left(x - \frac{2i-1}{2}\Delta\right)^2 f_x(x)\,dx + 2\int_{(M/2-1)\Delta}^{\infty} \left(x - \frac{2i-1}{2}\Delta\right)^2 f_x(x)\,dx$$

$$\frac{\delta\sigma_q^2}{\delta\Delta} = 0$$



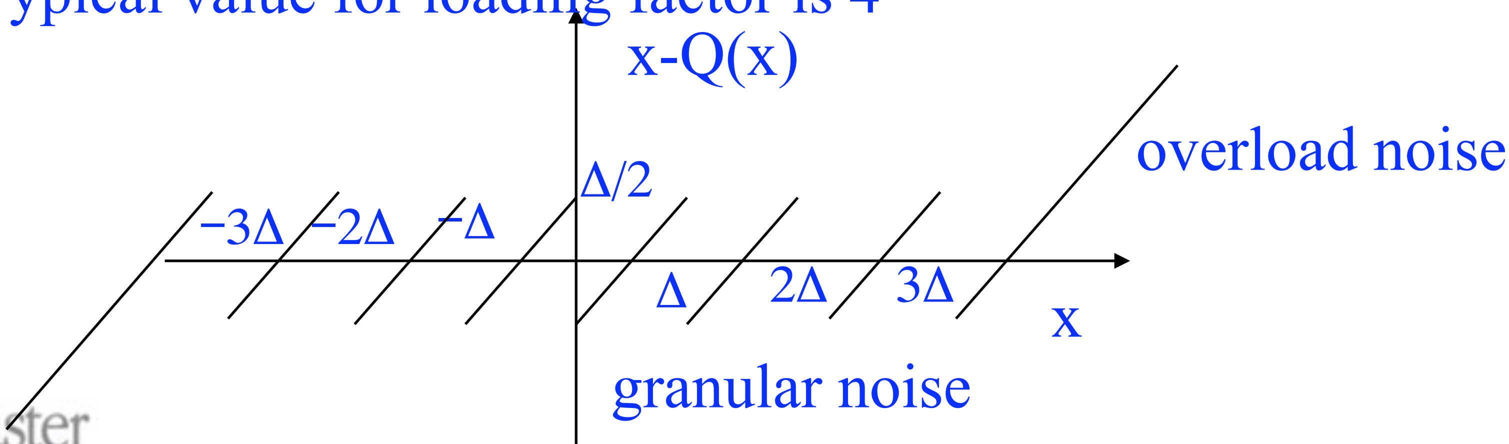
Uniform Quantizers for Non-uniform Source

- Given the $f_x(x)$ the value of step size can be calculated using numerical techniques
- Results are given for different values of M and different pdfs in table 9.3 of the book
- If input is unbounded, quantization error is no longer bounded
- In the inner intervals the error is bounded and is called granular error
- Unbounded error is called overload error



Uniform Quantizers for Non-uniform Source

- The pdf of most of non-uniform sources peaks at zero and decay away from the origin
- Overload probability is generally smaller than the probability of granular region
- Increasing step size will reduce overload error and increase granular error
- Loading factor is defined as the ratio of maximum value the input can take in the granular region to standard deviation
- Typical value for loading factor is 4



Adaptive Quantization

- Mismatch effects: occur when the pdf of the input signal changes. The reconstruction quality degrades.
- Quantizer adaptation:
  - FORWARD: A block of data is processed, mean and variance sent as side info.
  - BACKWARD: based on quantized values, no side info. Jayant quantizer.
- Forward: a delay is necessary, size of block of data: if block is too large the adaptation process may not capture the changes in the input statistics, if too small transmission of side information adds a significant overhead

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Adaptive Quantization

- If we study the input-output of a quantizer we can get an idea about the mismatch from the distribution of output values
- If Δ (quantizer step size) is smaller than what it should be, the input will fall in the outer levels of the quantizer an excessive number of times
- If Δ is larger than what it should be, the input will fall in the inner levels of the quantizer an excessive number of times
- Jayant quantizer: If the input falls in the outer levels, step size needs to be expanded, and if the input falls into inner levels, the step size needs to be reduced

Adaptive Quantization

- In Jayant quantizer expansion and contraction of the step size is accomplished by assigning a multiplier M_k to each interval.
- If the (n-1)th input falls in the kth interval, the step size to be used for the nth input is obtained by multiplying step size used for the (n-1)th input with M_k.
- Multiplier values for the inner levels in the quantizer are less than one and multiplier values for the outer levels are greater than one
- In math: $\Delta_n = M_{l(n-1)} \Delta_{n-1}$

Adaptive Quantization

- If input values are small for a period of time, the step size continues to shrink. In a finite precision system it would result in a value of zero.
- Solution: a minimum Δ_{min} is defined and the step size cannot go below this value
- Similarly to avoid too large values for the step size we define a Δ_{max}

Adaptive Quantization

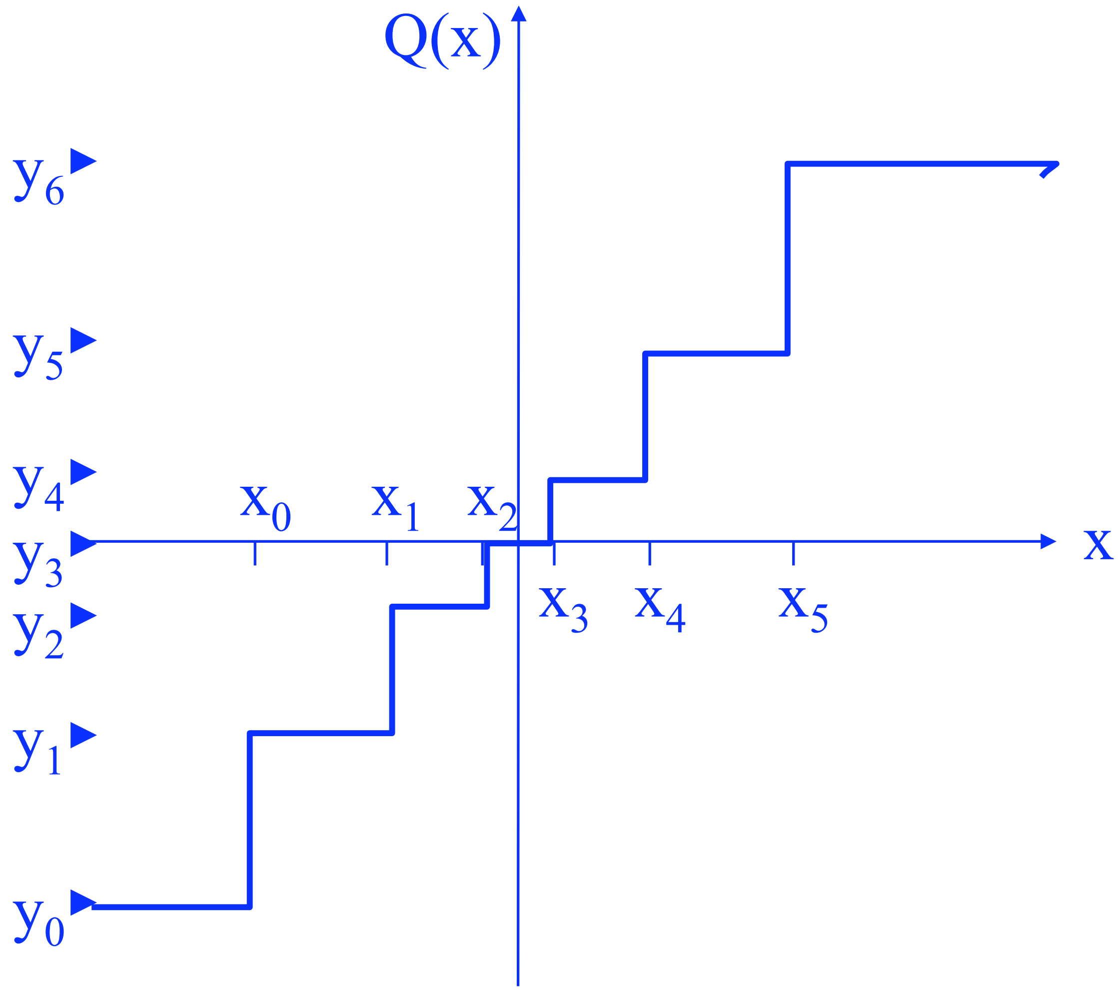
- How to choose the values of multipliers?
- Stability criteria: once the quantizer is matched to the input, the product of expansions and contractions are equal to one.

$$\prod_{k=0}^{M} M_k^{n_k} = 1 \Rightarrow \prod_{k=0}^{M} M_k^{n_k/N} = 1$$

$$\prod_{k=0}^{M} M_k^{P_k} = 1$$

$$M = \gamma^{l_k} \Rightarrow \prod_{k=0}^{M} \gamma^{l_k P_k} = 1 \Rightarrow \sum_{k=0}^{M} l_k P_k = 0$$

Non-uniform Scalar Quantizers



Non-uniform Quantizers

- Non uniform quantizers attempt to decrease the average distortion by assigning more levels to more probable regions.
- For given M and input pdf, we need to choose $\{b_i\}$ and $\{y_i\}$ to minimize the distortion

$$\sigma^2 = \sum_{i=1}^{M} \int_{b_{i-1}}^{b_i} (x - y_i)^2 f_X(x)\,dx$$

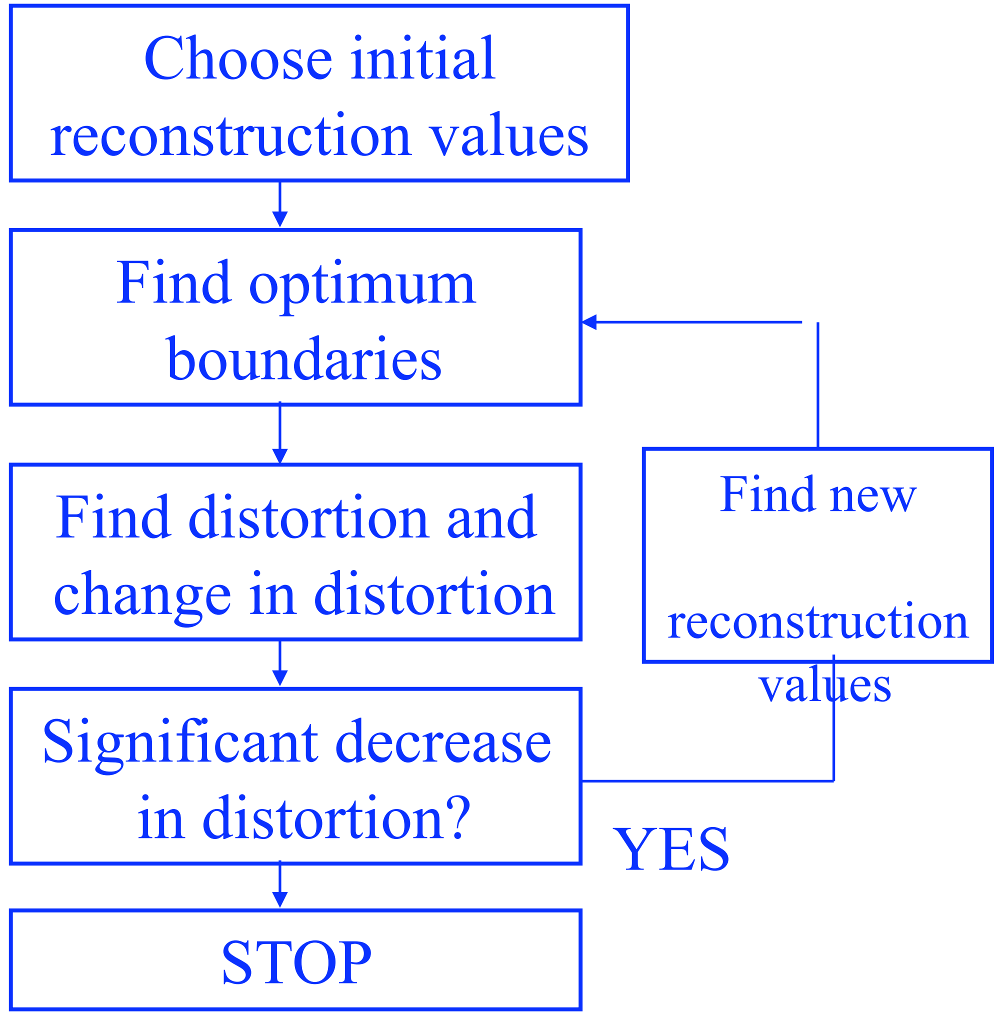
$$y_i = \frac{\int_{b_{i-1}}^{b_i} x f_X(x)\,dx}{\int_{b_{i-1}}^{b_i} f_X(x)\,dx} \qquad b_{i-1} = \frac{y_{i-1} + y_i}{2}$$

Non-uniform Quantizers

- The optimum value of reconstruction level depends on the boundary and the boundary depends on the reconstruction levels.
- Instead, it is easier to find these optimality conditions:
  - For a given partition (encoder), what is the optimum codebook (decoder)?
  - For a given codebook (decoder), what is the optimum partition (encoder)?

The Lloyd Algorithm

- It is difficult to solve both sets of equations analytically.
- An iterative algorithm known as the Lloyd algorithm solves the problem by iteratively optimizing the encoder and decoder until both conditions are met with sufficient accuracy.



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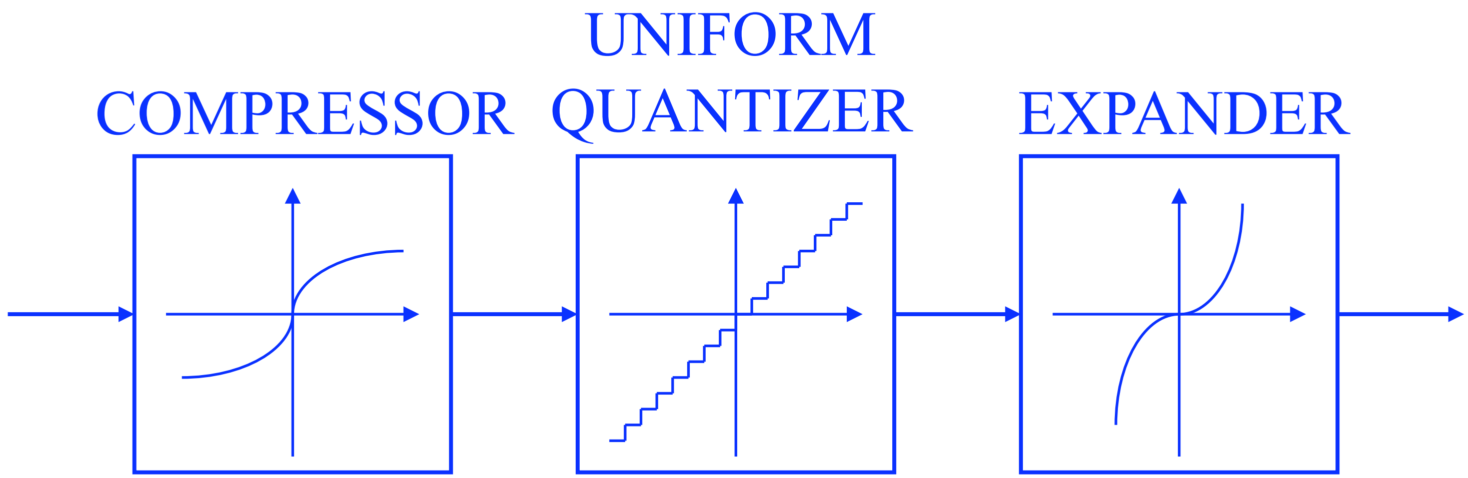
Companded Quantization



- Instead of making the step size small for intervals in which the input lies with high probability, make these intervals large and use a uniform quantizer
- Equivalent to the a non-uniform quantizer.
- Example: the μ-law compander:

$$c(x) = x_{max} \frac{\ln(1 + \mu \frac{|x|}{x_{max}})}{\ln(1 + \mu)} \operatorname{sign}(x)$$

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Entropy-Constrained Quantization

- Two approaches: 1) Keep the design of quantizer the same and entropy code the quantization output 2) Take into account the the selection of decision boundaries will affect the rate
- Joint optimization: entropy-constrained optimization. Minimize distortion subject to the constraint

$$H = \sum_i P_i \log P_i \le R$$

$$P_i = \int_{b_{i-1}}^{b_i} f_x(x)dx$$

$$\ln \frac{P_{k+1}}{P_k} = \lambda(y_{k+1} - y_k)(y_{k+1} + y_k - 2b_k)$$

$$y_j = \frac{\int_{b_{j-1}}^{b_j} x f_x(x)dx}{\int_{b_{j-1}}^{b_j} f_x(x)dx}$$

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High-Rate Optimum Quantization

$$b_k = \frac{y_{k+1} + y_k}{2} - \frac{1}{2\lambda(y_{k+1} - y_k)} \ln \frac{P_{k+1}}{P_k}$$

$$y_j = \frac{\int_{b_{j-1}}^{b_j} x f_x(x)\,dx}{\int_{b_{j-1}}^{b_j} f_x(x)\,dx}$$

- The above iterative method is called Generalized Lloyd algorithm.
- At high rates, the optimum entropy-constrained quantizer is the uniform quantizer!
- At high rates, $\mathrm{H(Q(X)) = h(X) - \log \frac{2x_{max}}{N}}$