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Note

On explicit formulas for bandwidth and antibandwidth of hypercubes

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ABSTRACT

The Hales numbered n-dimensional hypercube exhibits interesting recursive structures in n. These structures lead to a very simple proof of the well-known bandwidth formula for hypercubes proposed by Harper, whose proof was thought to be surprisingly difficult. Harper also proposed an optimal numbering for a related problem called the antibandwidth of hypercubes. In a recent publication, Raspaud et al. approximated the hypercube antibandwidth up to the third-order term. In this paper, we find the exact value in light of the above recursive structures.

1. Introduction

In the field of graph labeling [4,11], the problem of graph bandwidth has been extensively studied [2,3,7] and has found many applications such as parallel computations, VLSI circuit design, etc. In this paper we are particularly interested in the bandwidth and antibandwidth of hypercubes. The study of their values can guide the design of communication codes for error resilient transmission of signals over lossy networks such as the Internet [12].

First, we restate the definitions of vertex numbering and graph bandwidth, most of which are adopted from Harper's book [5]. A *numbering* of a vertex set V is any function $\eta : V \to \{1, 2, \ldots, |V|\}$, which is one-to-one (and therefore onto). A numbering η uniquely determines a total order, $\leq_\eta$, on V as follows: $u \leq_\eta v$ if $\eta(u) < \eta(v)$ or $\eta(u) = \eta(v)$. Conversely, a total order defined on V uniquely determines a numbering of the graph.

The *bandwidth of a numbering* η of a graph $G = (V, E)$ is the maximum difference

$$bw(\eta) = \max_{\{u,v\} \in E} |\eta(u) - \eta(v)|. \tag{1}$$

The *bandwidth of a graph G* is the minimum bandwidth over all numberings, η, of G, i.e.

$$bw(G) = \min_{\eta} bw(\eta). \tag{2}$$

Another vertex numbering problem related to graph bandwidth is what we call the antibandwidth problem. It is posed by reversing the objective of vertex numbering in that we now want to maximize the minimum distance between any adjacent pair of vertices. The antibandwidth problem of a graph $G = (V, E)$ is defined as

$$abw(G) = \max_{\eta} \min_{\{u,v\} \in E} |\eta(u) - \eta(v)|. \tag{3}$$

The antibandwidth problem was previously studied by Leung et al. in [8]. In [9], the graph antibandwidth problem was investigated by Miller and Pritikin under the name of *separation number*. Some antibandwidth related problems were also studied, such as the edge-bandwidth [1] and the cyclic antibandwidth [10].

For hypercubes, the bandwidth and antibandwidth problems are also well studied. In [6], Harper proposes a class of bandwidth achieving numberings for the hypercube and presents the exact bandwidth value without a proof. One such

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bandwidth achieving numbering is the Hales numbering defined in [5]. In Section 2, we will highlight the recursive structure of the Hales numbering, and propose a simple proof of the bandwidth formula in Section 3 based on this property.

In [1], a tight bound of the antibandwidth of the hypercube is given. Raspaud et al. further approximate the antibandwidth value of the hypercube up to the third-order term in [10]. However, in his classical 1966 paper [6], Harper has presented the antibandwidth achieving numbering for the hypercube, which is a variation of the Hales numbering. With the recursive structure of the Hales numbering, we determine in Section 4 the exact value of the antibandwidth for the hypercube, which is a new result in the literature.

2. Recursive structure of the Hales numbering

The graph of the n-dimensional cube, Q_n, has vertex set $\{0, 1\}^n$, the n-fold Cartesian product of $\{0, 1\}$. Thus $|V_{Q_n}| = 2^n$. Q_n has an edge between two vertices (n-tuples of 0s and 1s) if they differ in exactly one entry.

The *Hales order* [5], $\leq_H$, on V_{Q_n}, is defined by $u \leq_H v$ if

(1) $w(u) < w(v)$, or
(2) $w(u) = w(v)$ and u is greater than or equal to v in lexicographic order relative to the right-to-left order of the coordinates,

where $w(\cdot)$ is the Hamming weight of a vertex of Q_n. This total order determines a numbering, $H_n : V_{Q_n} \to \{1, 2, \dots, 2^n\}$, which is called *Hales numbering*. For $n = 4$, for instance, the Hales order is $0000 < 0001 < 0010 < 0100 < 1000 < 0011 < 0101 < 1001 < 0110 < 1010 < 1100 < 0111 < 1011 < 1101 < 1110 < 1111$.

The Hales numbering has a nice recursive structure as follows:

Theorem 1. *We define a $2^n \times n(0, 1)$-matrix S_n as*

$$S_n = \begin{bmatrix} A_0^{(n)} \\ A_1^{(n)} \\ \vdots \\ A_n^{(n)} \end{bmatrix}, \tag{4}$$

where $A_k^{(n)}$, $0 \leq k \leq n$, is an $\binom{n}{k} \times n(0, 1)$-matrix satisfying the following recursive formula

$$A_k^{(n)} = \begin{bmatrix} A_{k-1}^{(n-1)} & \mathbf{1} \\ A_k^{(n-1)} & \mathbf{0} \end{bmatrix}, \quad 1 \leq k \leq n-1, \tag{5}$$

where $\mathbf{0}$ and $\mathbf{1}$ are column vectors containing only 0s and 1s respectively. As the base case, we have $A_0^{(n)} = \mathbf{0}^T$ and $A_n^{(n)} = \mathbf{1}^T$. Then the row vectors of S_n, from top to bottom, are all vertices of Q_n in the increasing Hales order.

Proof. By definition, it is sufficient to show that the row vectors of $A_k^{(n)}$, $0 \leq k \leq n$, are all distinct vectors with Hamming weight k, which are sorted, from top to bottom, in the decreasing lexicographic order relative to the right-to-left order of the coordinates.

We prove by induction on n. The above assertion is trivially true for $n = 1$. Assume the assertion holds for $n - 1 \geq 1$. Now for n, $A_0^{(n)}$ is a vector of Hamming weight 0 and $A_n^{(n)}$ a vector of Hamming weight n, so the assertion trivially holds. For $1 \leq k \leq n-1$, the first $\binom{n-1}{k-1}$ vectors of $A_k^{(n)}$ are all distinct and have Hamming weight k by the induction assumption that all row vectors in $A_{k-1}^{(n-1)}$ are distinct and have Hamming weight $k - 1$. Further, these vectors are in the decreasing lexicographic order because they share the same rightmost bit and all vectors in $A_k^{(n-1)}$ are sorted. By the same argument the next $\binom{n-1}{k}$ vectors of $A_k^{(n)}$ are distinct, of Hamming weight k, and sorted in the decreasing lexicographic order as well. Combining the above facts and (5) concludes that the row vectors of $A_k^{(n)}$ are distinct, of Hamming weight k, and in the decreasing lexicographic order. □

3. Proof of the bandwidth formula for hypercubes

In [6] and [5, Corollary 4.3], Harper shows that the Hales numbering minimizes the bandwidth of the n-cube, i.e.

$$bw(H_n) = bw(Q_n). \tag{6}$$

He also gives the exact bandwidth value of the hypercube as follows.

Theorem 2 (*Harper, [5], Corollary 4.4*). *For the n-cube Q_n, we have*

$$bw(Q_n) = \sum_{m=0}^{n-1} \binom{m}{\lfloor \frac{m}{2} \rfloor}. \tag{7}$$

Although the above result has been known for forty years, no proof seemed to appear in the literature. In his recent book [5], Harper noted that the proof of the bandwidth formula of hypercube (Theorem 2) "is surprisingly difficult". In the following we present a quite simple proof, which is based on the recursive structure of Hales numbering as given in Theorem 1.

Proof. For any two vertices $u, v \in V_{Q_n}$, the condition $\{u, v\} \in E_{Q_n}$ implies that $|w(u)-w(v)| = 1$. Without loss of generality, let $w(v) = w(u) + 1$. If u is a row vector of $A_k^{(n)}$, denoted by $u \in A_k^{(n)},\ 0 \le k \le n-1$, then v must be a row vector of $A_{k+1}^{(n)}$, i.e. $v \in A_{k+1}^{(n)}$. If we define

$$d_k^{(n)} = \max_{\{u,v\}\in E_{Q_n}, u\in A_k^{(n)}, v\in A_{k+1}^{(n)}} |H_n(u) - H_n(v)|, \tag{8}$$

for $0 \le k \le n-1$, then the bandwidth of the Hales numbering becomes

$$bw(H_n) = \max_{0\le k\le n-1} d_k^{(n)}. \tag{9}$$

It is trivial that $d_0^{(n)} = d_{n-1}^{(n)} = n$. For $1 \le k \le n-2$, according to (5), $u \in A_k^{(n)}$ indicates two possibilities: $u \in [A_{k-1}^{(n-1)}\ \mathbf{1}]$ and $u \in [A_k^{(n-1)}\ \mathbf{0}]$. We define

$$d_{k,b}^{(n)} = \max_{\{u,v\}\in E_{Q_n}, u\in [A_{k-b}^{(n-1)}\ \mathbf{b}], v\in A_{k+1}^{(n)}} |H_n(u) - H_n(v)|, \tag{10}$$

for $1 \le k \le n-2$ and $b \in \{0, 1\}$, then it is obvious

$$d_k^{(n)} = \max\left\{d_{k,0}^{(n)}, d_{k,1}^{(n)}\right\}. \tag{11}$$

Now we derive a recursive formula for $d_{k,1}^{(n)},\ 1 \le k \le n-2$. Consider $\{u, v\} \in E_{Q_n}$ with $u \in [A_{k-1}^{(n-1)}\ \mathbf{1}]$ and $v \in A_{k+1}^{(n)}$. Using the recursive formula of (5), we can expand $A_{k+1}^{(n)}$ as follows

$$A_{k+1}^{(n)} = \begin{bmatrix} A_k^{(n-1)} & \mathbf{1} \\ A_{k+1}^{(n-1)} & \mathbf{0} \end{bmatrix}. \tag{12}$$

It is easy to see that v must be a row vector of $[A_k^{(n-1)}\ \mathbf{1}]$. Moreover, $u_{1:n-1} \in A_{k-1}^{(n-1)}$, $v_{1:n-1} \in A_k^{(n-1)}$ and $\{u_{1:n-1}, v_{1:n-1}\}$ is an edge in the $(n-1)$-dimensional hypercube, where $u_{1:n-1}$ denotes the vector formed by the leftmost $n-1$ bits of u. Since by Theorem 1, the rows of S_n, respectively S_{n-1}, from top to bottom, are the vertices of Q_n, respectively, Q_{n-1}, in increasing Hales order and because the number of rows in $[A_k^{(n-1)}\ \mathbf{0}]$ is $\binom{n-1}{k}$, we obtain

$$|H_n(u) - H_n(v)| = \binom{n-1}{k} + |H_{n-1}(u_{1:n-1}) - H_{n-1}(v_{1:n-1})|. \tag{13}$$

Since there is a one-to-one correspondence between the edges $\{u, v\} \in E_{Q_n}$ with $u \in [A_{k-1}^{(n-1)}\ \mathbf{1}]$, $v \in [A_k^{(n-1)}\ \mathbf{1}]$ and the edges $\{u_{1:n-1}, v_{1:n-1}\} \in E_{Q_{n-1}}$ with $u_{1:n-1} \in A_{k-1}^{(n-1)}$, $v_{1:n-1} \in A_k^{(n-1)}$, it follows further that

$$d_{k,1}^{(n)} = \binom{n-1}{k} + d_{k-1}^{(n-1)}, \quad 1 \le k \le n-2. \tag{14}$$

In order to derive a recursion for $d_{k,0}^{(n)},\ 1 \le k \le n-2$, consider now $\{u, v\} \in E_{Q_n}$ with $u \in [A_k^{(n-1)}\ \mathbf{0}]$ and $v \in A_{k+1}^{(n)}$. Then v could be either in $[A_k^{(n-1)}\ \mathbf{1}]$ or in $[A_{k+1}^{(n-1)}\ \mathbf{0}]$. But since the rows of S_n are in increasing Hales order, from top to bottom, it suffices to consider only $v \in [A_{k+1}^{(n-1)}\ \mathbf{0}]$ in the maximization of (10) for $d_{k,0}^{(n)}$. Further, similarly to the discussion for $d_{k,1}^{(n)}$, we obtain

$$d_{k,0}^{(n)} = \binom{n-1}{k} + d_k^{(n-1)}, \quad 1 \le k \le n-2. \tag{15}$$

Making further the convention that $d_{-1}^{(n)} = 0$ and $d_n^{(n)} = 0$ for any $n \ge 1$, relations (11), (14), (15) and the fact $d_0^{(n)} = d_{n-1}^{(n)} = n$ imply that

$$d_k^{(n)} = \binom{n-1}{k} + \max\left\{d_{k-1}^{(n-1)}, d_k^{(n-1)}\right\}, \quad 0 \le k \le n-1. \tag{16}$$

In order to complete the proof of the theorem we next prove the following assertion.
Assertion. For any $n \geq 1$, we have

$$d_k^{(n)} \leq \sum_{m=0}^{n-1} \binom{m}{\lfloor \frac{m}{2} \rfloor}, \quad 0 \leq k \leq n-1. \tag{17}$$

Moreover, (17) holds with equality if $k = \lfloor \frac{n}{2} \rfloor$.

Proof of Assertion. We prove the assertion by induction on n. The base case $n = 1$ follows immediately since $d_0^{(1)} = 1 = \binom{0}{0}$ and $0 = \lfloor \frac{1-1}{2} \rfloor$. Assume now that the assertion is true for $n - 1 \geq 1$. Hence,

$$d_k^{(n-1)} \leq \sum_{m=0}^{n-2} \binom{m}{\lfloor \frac{m}{2} \rfloor}, \quad 0 \leq k \leq n-2, \tag{18}$$

with equality when $k = \lfloor \frac{n-1}{2} \rfloor$. Consider k, $1 \leq k \leq n-1$. Using (16) together with the induction hypothesis we obtain that

$$\begin{aligned} d_k^{(n)} &= \binom{n-1}{k} + \max\left\{d_{k-1}^{(n-1)}, d_k^{(n-1)}\right\} \\ &\leq \binom{n-1}{\lfloor \frac{n}{2} \rfloor} + \sum_{m=0}^{n-2} \binom{m}{\lfloor \frac{m}{2} \rfloor} \\ &= \sum_{m=0}^{n-1} \binom{m}{\lfloor \frac{m}{2} \rfloor}, \end{aligned} \tag{19}$$

which proves inequality (17). Assume now that $k = \lfloor \frac{n}{2} \rfloor$. Then when n is even, $k-1 = \lfloor \frac{n}{2} \rfloor - 1 = \lfloor \frac{n-1}{2} \rfloor$, and $d_{k-1}^{(n-1)}$ achieves equality in (18). When n is odd, $k = \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n-1}{2} \rfloor$, hence $d_k^{(n-1)}$ also achieves equality in (18). Both cases lead to equality in (19), thus completing the proof of the assertion. Now the conclusion of Theorem 2 follows immediately. □

4. The antibandwidth formula of hypercubes

According to Harper [6], an antibandwidth achieving numbering for the hypercube can be obtained as follows. First number the vertices with even Hamming weights in increasing Hales order, and then number the vertices with odd Hamming weights, also in the increasing Hales order. Let us denote this numbering by $\tilde{H_n}(\cdot)$. An example of such a numbering for $n = 4$ is $0000 < 0011 < 0101 < 1001 < 0110 < 1010 < 1100 < 1111 < 0001 < 0010 < 0100 < 1000 < 0111 < 1011 < 1101 < 1110$. In this section, we provide a closed-form formula for the solution of the antibandwidth problem on n-cubes, which is a new result.

Theorem 3. *For the numbering $\tilde{H_n}$, we have*

$$abw(\tilde{H_n}) = 2^{n-1} - \sum_{m=0}^{n-2} \binom{m}{\lfloor \frac{m}{2} \rfloor}. \tag{20}$$

Proof. Let $\tilde{S_n}$ be the $2^n \times n(0, 1)$-matrix whose rows from top to bottom are the vertices of Q_n in increasing order of $\tilde{H_n}$. Using the recursive structure in (5), we obtain

$$\tilde{S_n} = \begin{bmatrix} A_0^{(n)} \\ A_2^{(n)} \\ \vdots \\ A_{2\lfloor \frac{n}{2} \rfloor}^{(n)} \\ A_1^{(n)} \\ A_3^{(n)} \\ \vdots \\ A_{2\lfloor \frac{n-1}{2} \rfloor + 1}^{(n)} \end{bmatrix} = \begin{bmatrix} A_0^{(n-1)} & \mathbf{0} \\ A_1^{(n-1)} & \mathbf{1} \\ A_2^{(n-1)} & \mathbf{0} \\ \vdots & \\ A_{n-1}^{(n-1)} & b \\ A_0^{(n-1)} & \mathbf{1} \\ A_1^{(n-1)} & \mathbf{0} \\ A_2^{(n-1)} & \mathbf{1} \\ \vdots & \\ A_{n-1}^{(n-1)} & \bar{b} \end{bmatrix} = \begin{bmatrix} S_{n-1} & \mathbf{B} \\ S_{n-1} & \bar{\mathbf{B}} \end{bmatrix}, \tag{21}$$

where $b = 0$ if n is odd and $b = 1$ if n is even. We also use the relations $A_0^{(n)} = \mathbf{0}^T = [A_0^{(n-1)}\ \mathbf{0}]$ and $A_n^{(n)} = \mathbf{1}^T = [A_{n-1}^{(n-1)}\ \mathbf{1}]$. Note that $\mathbf{B}$ is a column vector of 2^{n-1} elements whose definition is straightforward and omitted here. Two observations regarding $\tilde{S_n}$ are crucial for the further development, namely:

(O1) the leftmost $n-1$ bits of the top (bottom) 2^{n-1} vertices form a Hales numbered hypercube in $n-1$ dimensions, i.e. S_{n-1} (see (4));
(O2) if u is the vertex on some row i, $i \le 2^{n-1}$, then the vertex on row $i + 2^{n-1}$ is u' obtained by flipping the last bit of u.

Consider now an edge $\{u, v\} \in E_{Q_n}$. Clearly the Hamming weights $w(u)$ and $w(v)$ have different parities. Assume without restricting the generality that $w(u)$ is odd and $w(v)$ is even, hence u is in the top half and v is in the bottom half of $\tilde{S_n}$. Then according to observation O2, if $v = u'$ we have

$$\tilde{H_n}(v) - \tilde{H_n}(u) = 2^{n-1} = 2^{n-1} + H_{n-1}(v_{1:n-1}) - H_{n-1}(u_{1:n-1}), \tag{22}$$

where the last equality is due to the fact that $v_{1:n-1} = u_{1:n-1}$. On the other hand, if $v \ne u'$, then

$$\begin{aligned}\tilde{H_n}(v) - \tilde{H_n}(u) &= \tilde{H_n}(u') - \tilde{H_n}(u) + \tilde{H_n}(v) - \tilde{H_n}(u')\\ &= 2^{n-1} + H_{n-1}(v_{1:n-1}) - H_{n-1}({u'}_{1:n-1})\\ &= 2^{n-1} + H_{n-1}(v_{1:n-1}) - H_{n-1}(u_{1:n-1}),\end{aligned} \tag{23}$$

where the second equality is due to O2 and O1 noting that both v and u' are in the bottom half of $\tilde{S_n}$. Moreover, the last equality in (23) follows from ${u'}_{1:n-1} = u_{1:n-1}$. Finally, by combining (22) and (23), we obtain

$$\begin{aligned}abw(Q_n) &= \min_{\{u,v\}\in E_{Q_n}} \left|\tilde{H_n}(v) - \tilde{H_n}(u)\right|\\ &= \min_{\{u,v\}\in E_{Q_n}} \left|2^{n-1} + H_{n-1}(u_{1:n-1}) - H_{n-1}(v_{1:n-1})\right|\\ &\overset{(a)}{=} \min_{\{u_{1:n-1},v_{1:n-1}\}\in E_{Q_{n-1}}} \left\{2^{n-1}, 2^{n-1} + H_{n-1}(u_{1:n-1}) - H_{n-1}(v_{1:n-1})\right\}\\ &\overset{(b)}{=} 2^{n-1} - \max_{\{u_{1:n-1},v_{1:n-1}\}\in E_{Q_{n-1}}} \{|H_{n-1}(v_{1:n-1}) - H_{n-1}(u_{1:n-1})|\}\\ &= 2^{n-1} - bw(Q_{n-1})\\ &= 2^{n-1} - \sum_{m=0}^{n-2} \binom{m}{\lfloor \frac{m}{2} \rfloor},\end{aligned} \tag{24}$$

where (a) is due to the fact that $2^{n-1} + H_{n-1}(u_{1:n-1}) - H_{n-1}(v_{1:n-1}) > 0$ and when $\{u, v\}$ varies over E_{Q_n}, $\{u_{1:n-1}, v_{1:n-1}\}$ varies over $E_{Q_{n-1}} \cup \{\{\nu, \nu\} | \nu \in Q_{n-1}\}$. Finally, (b) follows from the fact that the set of values $H_{n-1}(v_{1:n-1}) - H_{n-1}(u_{1:n-1})$ is symmetric relative to 0. □

Then based on Harper's result that $\tilde{H_n}$ is the antibandwidth achieving numbering, we have the following corollary.

Corollary 4.

$$abw(Q_n) = 2^{n-1} - \sum_{m=0}^{n-2} \binom{m}{\lfloor \frac{m}{2} \rfloor}. \tag{25}$$

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References

[1] T. Calamoneri, A. Massini, I. Vrt'o, New results on edge-bandwidth, Theoretical Computer Science 307 (3) (2003) 503–513.
[2] P.Z. Chinn, J. Chvátalová, A.K. Dewdney, N.E. Gibbs, The bandwidth problem for graphs and matrices — a survey, Journal of Graph Theory 6 (1982) 223–254.
[3] J. Diaz, J. Petit, M. Serna, A survey of graph layout problems, ACM Computing Surveys 34 (3) (2002) 313–356.
[4] J.A. Gallian, A dynamic survey of graph labeling, The Electronic Journal of Combinatorics 15 (DS6) (1998).
[5] L.H. Harper, Global Methods for Combinatorial Isoperimetric Problems, Cambridge University Press, 2004.

[6] L.H. Harper, Optimal numbering and isoperimetric problems on graphs, Journal of Combinatorial Theory 1 (1966) 385–393.
[7] Y.-L. Lai, K. Williams, A survey of solved problems and applications on bandwidth, edgesum, and profile of graphs, Journal of Graph Theory 31 (2) (1999) 75–94.
[8] J. Leung, O. Vornberger, J. Witthoff, On some variants of the bandwidth minimization problem, SIAM Journal on Computing 13 (1984) 650.
[9] Z. Miller, D. Pritikin, On the separation number of a graph, Networks 19 (1989) 651–666.
[10] A. Raspaud, H. Schröder, O. Sýkora, L. Torok, I. Vrt'o, Antibandwidth and cyclic antibandwidth of meshes and hypercubes, Discrete Mathematics, 2008, (in press) doi:10.1016/j.disc.2007.12.058.
[11] A. Rosa, On certain valuations of the vertices of a graph, in: Proc. Inter. Symp., Rome, July 1966, Theory of Graphs (1967) 349–355.
[12] X. Wang, X. Wu, S. Dumitrescu, On optimal index assignment for MAP decoding of Markov sequences, in: Proc. IEEE Int. Symp. on Information Theory, Seattle, Washington, 2006, pp. 2314–2318.