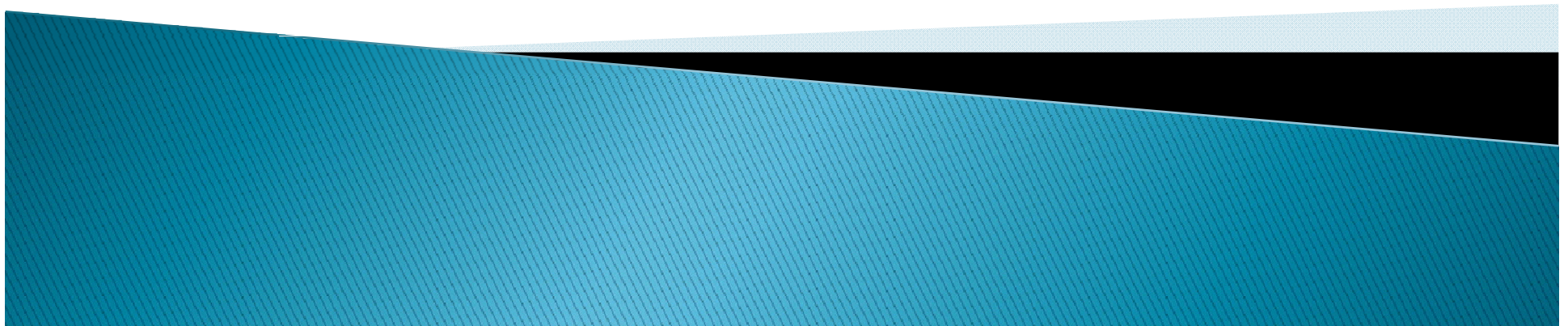


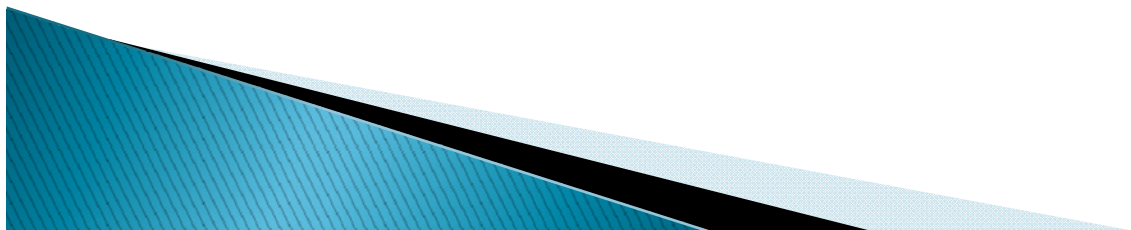
Tutorial for 3SK3

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Question 1

- ▶ A number of matrices are defined as

$$[A] = \begin{bmatrix} 4 & 7 \\ 1 & 2 \\ 5 & 6 \end{bmatrix} \quad [B] = \begin{bmatrix} 4 & 3 & 7 \\ 1 & 2 & 6 \\ 2 & 0 & 4 \end{bmatrix} \quad [C] = \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix}$$

$$[D] = \begin{bmatrix} 9 & 4 & 3 & -6 \\ 2 & -1 & 7 & 5 \end{bmatrix} \quad [E] = \begin{bmatrix} 1 & 5 & 9 \\ 7 & 2 & 3 \\ 4 & 0 & 6 \end{bmatrix}$$

$$[E] = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 7 & 3 \end{bmatrix} \quad [G] = [7 \ 6 \ 4]$$

Question 1

(a)

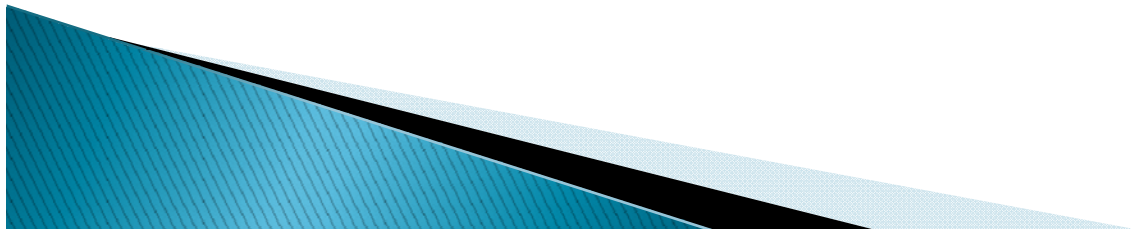
$$\begin{array}{llll} [A] = 3 \times 2 & [B] = 3 \times 3 & [C] = 3 \times 1 & [D] = 2 \times 4 \\ [E] = 3 \times 3 & [F] = 2 \times 3 & [G] = 1 \times 3 & \end{array}$$

(b)

Square: $[B]$ and $[E]$

Column: $[C]$

Row: $[G]$

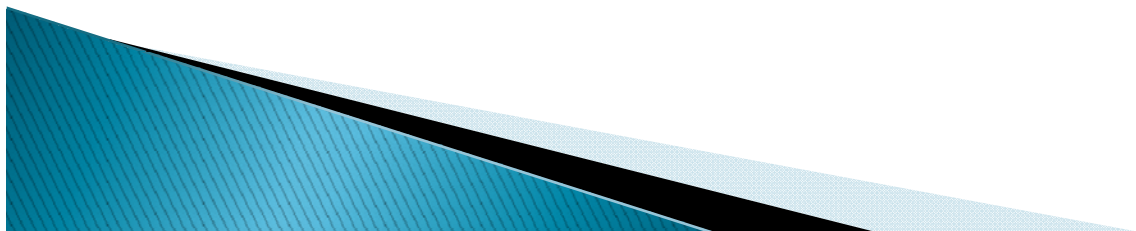


Question 1

(c) $a_{12} = 7$ $b_{23} = 7$ $d_{32} = \text{does not exist}$
 $e_{22} = 2$ $f_{12} = 0$ $g_{12} = 6$

(d) (1) $[E] + [B] = \begin{bmatrix} 5 & 8 & 15 \\ 8 & 4 & 10 \\ 6 & 0 & 10 \end{bmatrix}$ (2) $[A] + [F] = \text{not possible}$

(3) $[B] - [E] = \begin{bmatrix} 3 & -2 & -1 \\ -6 & 0 & 4 \\ -2 & 0 & -2 \end{bmatrix}$ (4) $7[B] = \begin{bmatrix} 28 & 21 & 49 \\ 7 & 14 & 49 \\ 14 & 0 & 28 \end{bmatrix}$



Question 1

$$(5) [E] \times [B] = \begin{bmatrix} 25 & 13 & 74 \\ 36 & 25 & 75 \\ 28 & 12 & 52 \end{bmatrix}$$

$$(7) [B] \times [A] = \begin{bmatrix} 54 & 76 \\ 41 & 53 \\ 28 & 38 \end{bmatrix}$$

$$(9) [A] \times [C] = \text{not possible}$$

$$(11) [E]^T [E] = \begin{bmatrix} 66 & 19 & 53 \\ 19 & 29 & 46 \\ 53 & 46 & 109 \end{bmatrix}$$

$$(6) \{C\}^T = [3 \quad 6 \quad 1]$$

$$(8) \{D\}^T = \begin{bmatrix} 9 & 2 \\ 4 & -1 \\ 3 & 7 \\ -6 & 5 \end{bmatrix}$$

$$(10) [I] \times [B] = [B]$$

$$(12) [C]^T [C] = 46$$



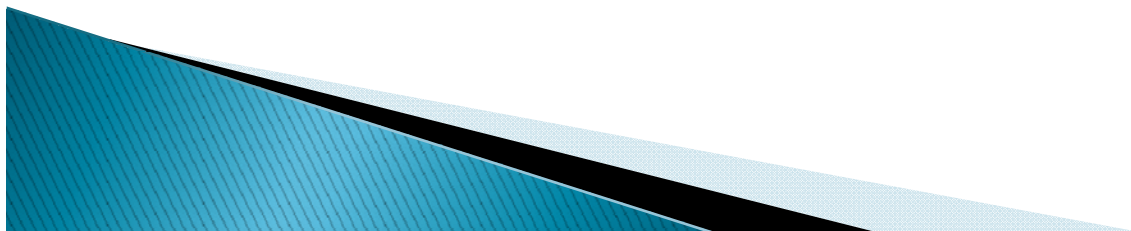
Question 2 (Q 9.7)

- ▶ Given the equations

$$0.5x_1 - x_2 = -9.5$$

$$1.02x_1 - 2x_2 = -18.8$$

- a) solve graphically
- b) Compute the determinant
- c) On the basis of a) and b), what would you expect regarding the system's condition.
- d) Solve by the elimination of unknowns



Question 2 (Q 9.7)

- ▶ (a) The equations can be rearranged into a format for plotting x_2 versus x_1 :

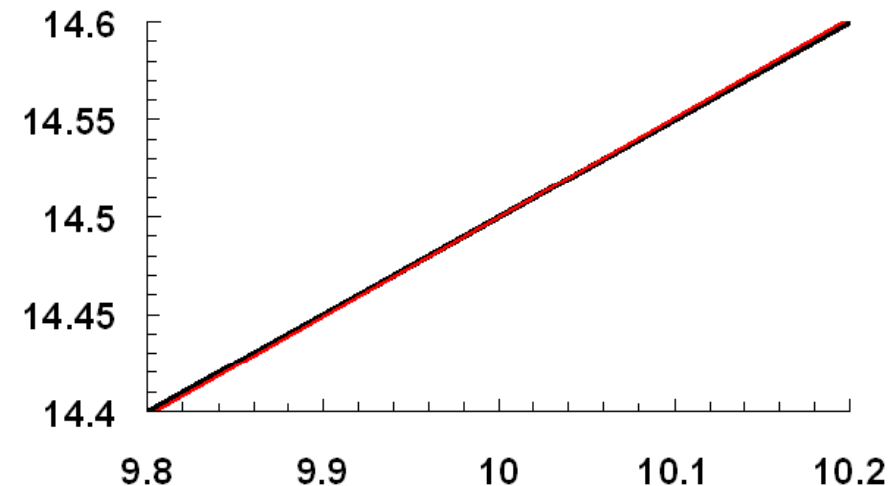
$$x_2 = 9.5 + 0.5x_1$$

$$x_2 = 9.4 + 0.51x_1$$

The solution is

$$x_1 = 10, x_2 = 14.5$$

Notice that the lines have very similar slopes.

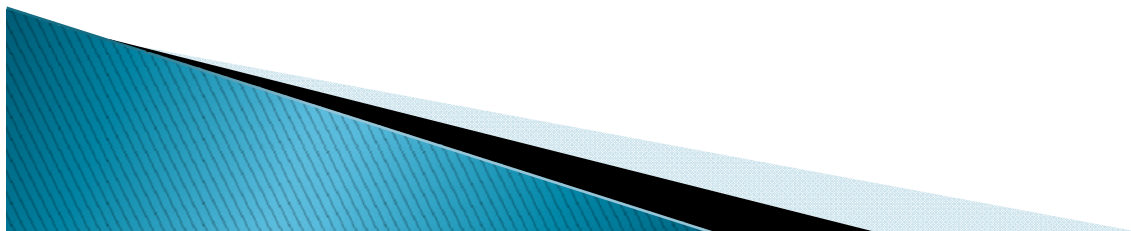


Question 2 (Q 9.7)

- ▶ b) The determinant can be computed as

$$D = 0.5(-2) - (-1)1.02 = 0.02$$

- ▶ c) The plot and the low value of the determinant both suggest that the system is ill-conditioned.



Question 2 (Q 9.7)

- ▶ (d) Using Eqs. (9.10) and (9.11) yields

$$x_1 = \frac{-9.5(-2) - (-1)(-18.8)}{-0.02} = -10$$

$$x_2 = \frac{0.52(-18.8) - (-9.5)1.02}{-0.02} = 4.3$$

The ill-conditioned nature of the system is illustrated by the fact that a small change in one of the coefficients results in a huge change in the results.



LU decomposition

--without pivoting strategy

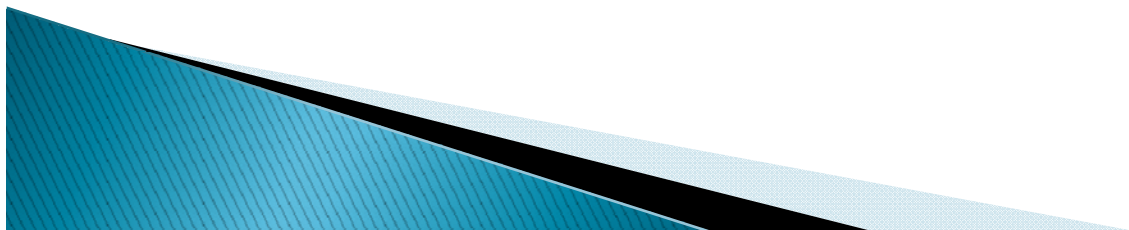
$$2x_1 + x_2 - x_3 + 2x_4 = 5$$

$$4x_1 + 5x_2 - 3x_3 + 6x_4 = 9$$

$$-2x_1 + 5x_2 - 2x_3 + 6x_4 = 4$$

$$4x_1 + 11x_2 - 4x_3 + 8x_4 = 2$$

$$[A|b] = \left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 5 \\ 4 & 5 & -3 & 6 & 9 \\ -2 & 5 & -2 & 6 & 4 \\ 4 & 11 & -4 & 8 & 2 \end{array} \right]$$

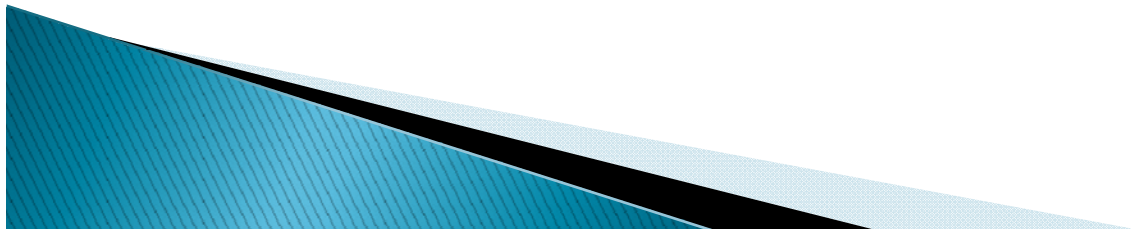


LU decomposition

--without pivoting strategy

- ▶ When eliminate a_{21} , a_{31} , a_{41} , we need to use $m_{2,1} = 2$, $m_{3,1} = -1$, $m_{4,1} = 2$ times a_{11} , then we will get

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 5 \\ 0 & 3 & -1 & 2 & -1 \\ 0 & 6 & -3 & 8 & 9 \\ 0 & 9 & -2 & 4 & -8 \end{array} \right]$$

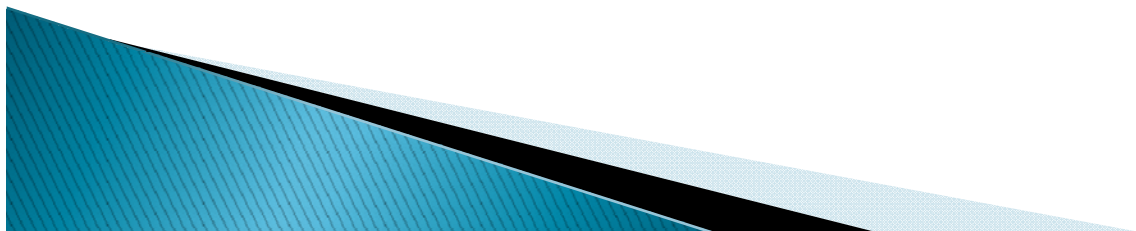


LU decomposition

--without pivoting strategy

- ▶ When eliminate a_{32} , a_{42} , we need to use $m_{3,2} = 2$, $m_{4,2} = 3$ times a_{22} , then we will get

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 5 \\ 0 & 3 & -1 & 2 & -1 \\ 0 & 0 & -1 & 4 & 11 \\ 0 & 0 & 1 & -2 & -5 \end{array} \right]$$



LU decomposition

--without pivoting strategy

- ▶ When eliminate a_{43} , we need to use $m_{4,3} = -1$ times a_{33} , then we will get

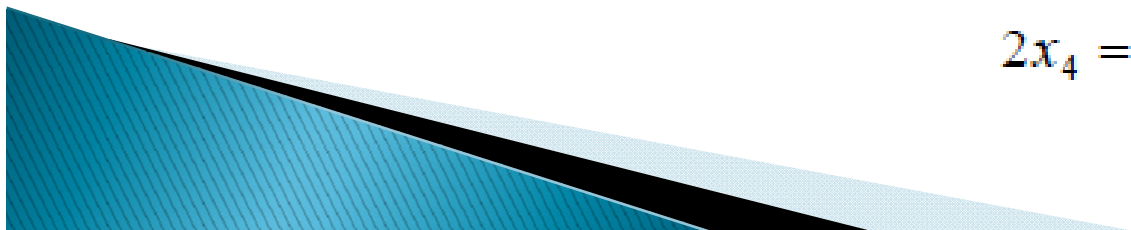
$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 5 \\ 0 & 3 & -1 & 2 & -1 \\ 0 & 0 & -1 & 4 & 11 \\ 0 & 0 & 0 & 2 & 6 \end{array} \right]$$

$$2x_1 + x_2 - x_3 + 2x_4 = 5$$

$$3x_2 - x_3 + 2x_4 = -1$$

$$-x_3 + 4x_4 = 11$$

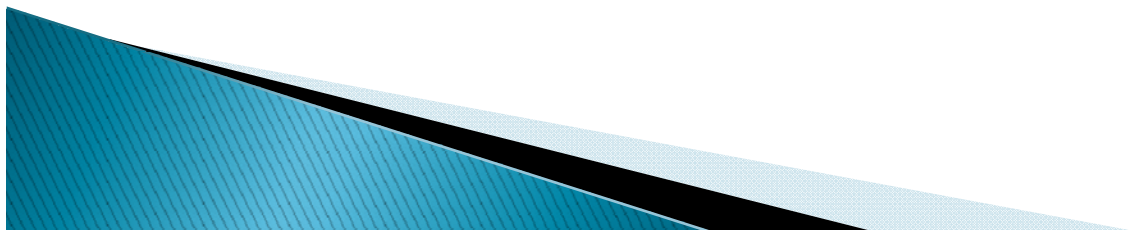
$$2x_4 = 6$$



LU decomposition

--without pivoting strategy

$$U = \begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ m_{2,1} & 1 & 0 & 0 \\ m_{3,1} & m_{3,2} & 1 & 0 \\ m_{4,1} & m_{4,2} & m_{4,3} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 3 & -1 & 1 \end{bmatrix}$$



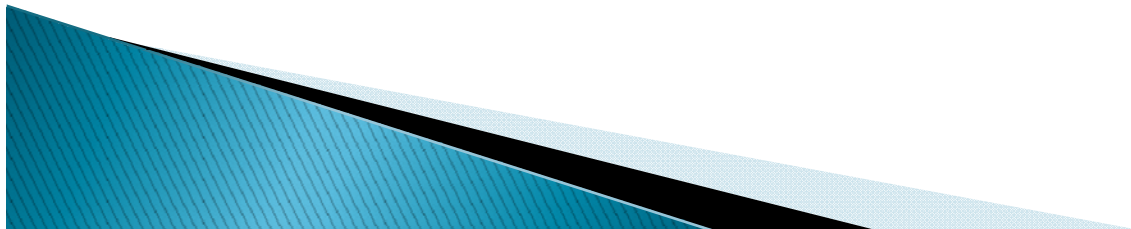
Question 3 (Q 10.6)

- ▶ Use LU decomposition to determine the matrix inverse for the following system. **Do not use a pivoting strategy**, and check your results by verifying that $[A][A]^{-1} = [I]$

$$10x_1 + 2x_2 - x_3 = 27$$

$$-3x_1 - 6x_2 + 2x_3 = -61.5$$

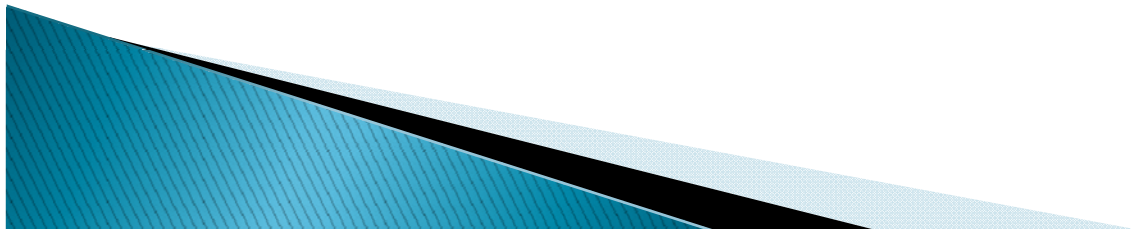
$$x_1 + x_2 + 5x_3 = -21.5$$



Question 3 (Q 10.6)

- ▶ First, we compute the LU decomposition. The coefficient a_{21} is eliminated by multiplying row 1 by $f_{21} = -3/10 = -0.3$ and subtracting the result from row 2. a_{31} is eliminated by multiplying row 1 by $f_{31} = 1/10 = 0.1$ and subtracting the result from row 3. The factors f_{21} and f_{31} can be stored in a_{21} and a_{31} .

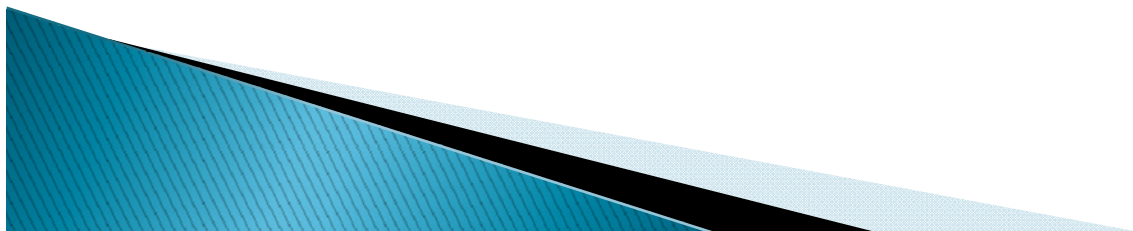
$$\begin{bmatrix} 10 & 2 & -1 \\ -0.3 & -5.4 & 1.7 \\ 0.1 & 0.8 & 5.1 \end{bmatrix}$$



Question 3 (Q 10.6)

- ▶ a_{32} is eliminated by multiplying row 2 by $f_{32} = 0.8/(-5.4) = -0.148148$ and subtracting the result from row 3. The factor f_{32} can be stored in a_{32} .

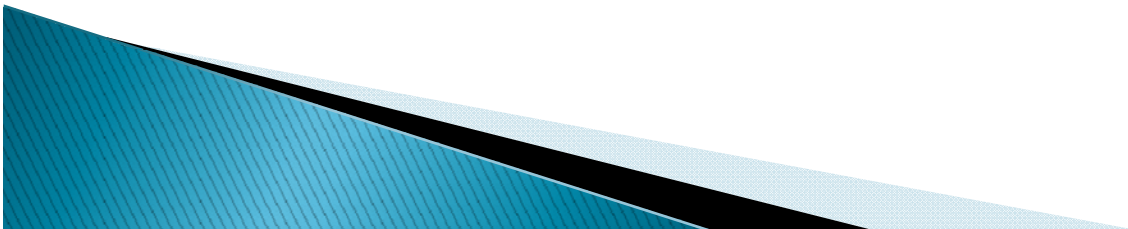
$$\begin{bmatrix} 10 & 2 & -1 \\ -0.3 & -5.4 & 1.7 \\ 0.1 & -0.148148 & 5.351852 \end{bmatrix}$$



Question 3 (Q 10.6)

- ▶ Therefore, the LU decomposition is

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.148148 & 1 \end{bmatrix} \quad [U] = \begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix}$$



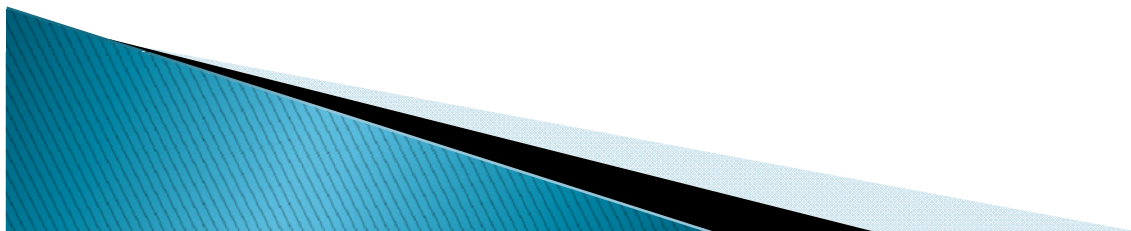
Question 3 (Q 10.6)

- ▶ The first column of the inverse can be computed by using $[L]\{D\} = \{B\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.148148 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

This can be solved for $d_1 = 1$, $d_2 = 0.3$, and $d_3 = -0.055556$. Then, we can implement back substitution

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0.3 \\ -0.055556 \end{Bmatrix}$$




Question 3 (Q 10.6)

to yield the first column of the inverse

$$\{X\} = \begin{Bmatrix} 0.110727 \\ -0.058824 \\ -0.0103806 \end{Bmatrix}$$

For the second column use $\{B\}^T = \{0 \ 1 \ 0\}$ which gives $\{D\}^T = \{0 \ 1 \ 0.148148\}$. Back substitution then gives $\{X\}^T = \{0.038062 \ -0.176471 \ 0.027682\}$.

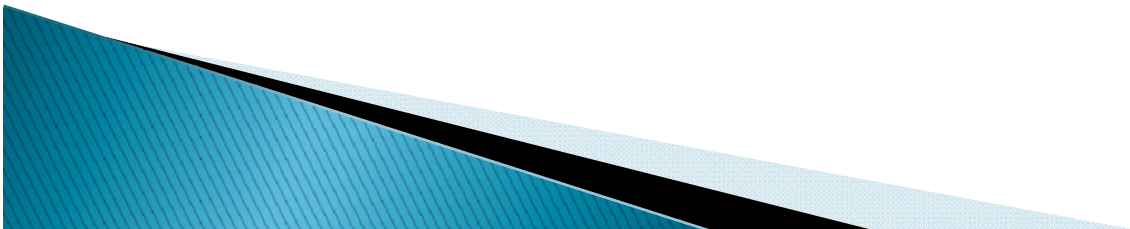
For the third column use $\{B\}^T = \{0 \ 0 \ 1\}$ which gives $\{D\}^T = \{0 \ 0 \ 1\}$. Back substitution then gives $\{X\}^T = \{0.00692 \ 0.058824 \ 0.186851\}$.



Question 3 (Q 10.6)

Therefore, the matrix inverse is

$$[A]^{-1} = \begin{bmatrix} 0.110727 & 0.038062 & 0.006920 \\ -0.058824 & -0.176471 & 0.058824 \\ -0.010381 & 0.027682 & 0.186851 \end{bmatrix}$$

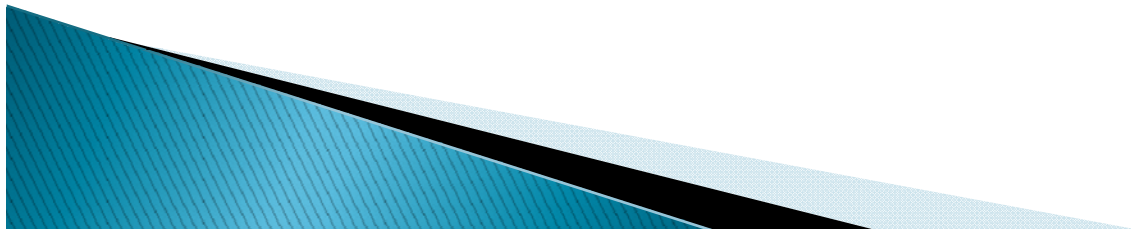


Question 3 (Q 10.6)

We can verify that this is correct by multiplying $[A][A]^{-1}$ to yield the identity matrix. For example, using MATLAB,

```
>> A=[10 2 -1;-3 -6 2;1 1 5];  
>> AI=[0.110727 0.038062 0.006920;  
-0.058824 -0.176471 0.058824;  
-0.010381 0.027682 0.186851];  
>> A*AI
```

```
ans =  
    1.0000    -0.0000   -0.0000  
    0.0000     1.0000   -0.0000  
   -0.0000     0.0000     1.0000
```



Question 4 (Q 10.15)

- ▶ a) Determine the condition number for the following system using the row-sum norm. Do not normalize the system.

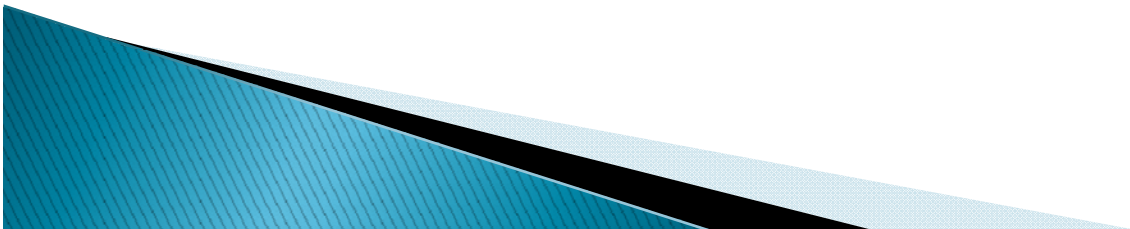
$$\begin{bmatrix} 1 & 4 & 9 & 16 & 25 \\ 4 & 9 & 16 & 25 & 36 \\ 9 & 16 & 25 & 36 & 49 \\ 16 & 25 & 36 & 49 & 64 \\ 25 & 36 & 49 & 64 & 81 \end{bmatrix}$$

How many digits of precision will be lost due to ill-conditioning?



Question 4 (Q 10.15)

- ▶ b) Repeat (a), but scale the matrix by making the maximum element in each row equal to one.



Question 4 (Q 10.15)

- ▶ (a) In order to compute the row-sum norm, we can determine the sum of the absolute values of each of the rows:

						row sums ↓
1	4	9	16	25		55
4	9	16	25	36		90
9	16	25	36	49		135
16	25	36	49	64		190
25	36	49	64	81		255

Therefore, $\|A\|_{\infty} = 255$. The matrix inverse can then be computed.



Question 4 (Q 10.15)

For example, using MATLAB,

```
>> A=[1 4 9 16 25;  
4 9 16 25 36;  
9 16 25 36 49;  
16 25 36 49 64;  
25 36 49 64 81];  
>> AI=inv(A)
```

Warning: Matrix is close to singular or badly scaled.

Results may be inaccurate. RCOND = 9.944077e-019.

AI =

1.0e+015 *

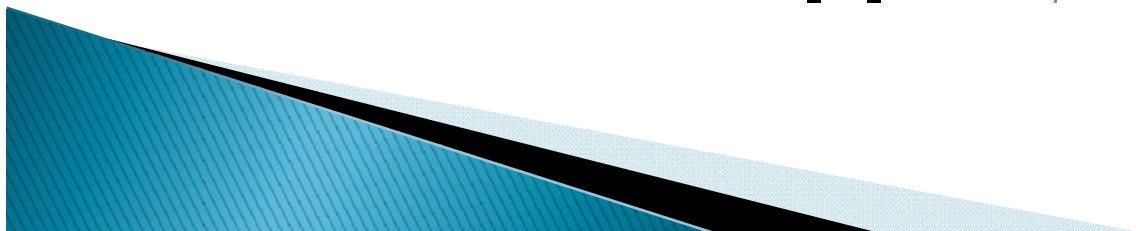
-0.2800	0.6573	-0.2919	-0.2681	0.1827
0.5211	-1.3275	0.8562	0.1859	-0.2357
0.1168	0.0389	-0.8173	1.0508	-0.3892
-0.6767	1.2756	0.2335	-1.5870	0.7546
0.3189	-0.6443	0.0195	0.6184	-0.3124



Question 4 (Q 10.15)

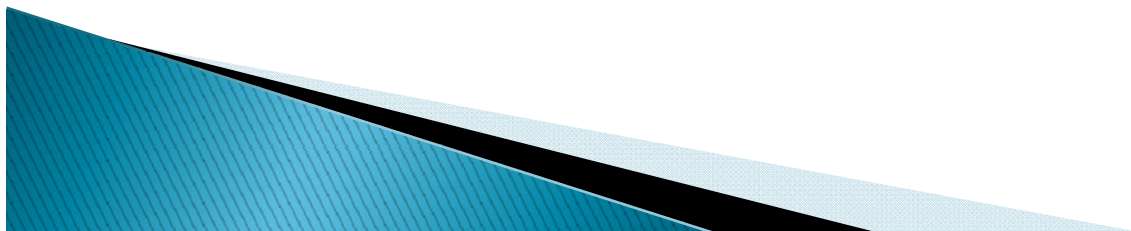
Notice that MATLAB alerts us that the matrix is ill-conditioned. This is also strongly suggested by the fact that the elements are so large.

The row-sum norm can then be computed by determining the sum of the absolute values of each of the rows. The result is $\|A^{-1}\|_{\infty} = 4.5274 \times 10^{15}$. Therefore, the condition number can be computed as $\text{Cond}[A] = 255(4.5274 \times 10^{15}) = 1.1545 \times 10^{18}$.



Question 4 (Q 10.15)

- ▶ This corresponds to $\log_{10}(1.1545 \times 10^{18}) = 18.06$ suspect digits. Thus, the suspect digits are more than the number of significant digits for the double precision representation used in MATLAB (15–16 digits). Consequently, we can conclude that this matrix is highly ill-conditioned.



Question 4 (Q 10.15)

(b) First, the matrix is scaled. For example, using MATLAB,

```
>> A=[1/25 4/25 9/25 16/25 25/25;  
4/36 9/36 16/36 25/26 36/36;  
9/49 16/49 25/49 36/49 49/49;  
16/64 25/64 36/64 49/64 64/64;  
25/81 36/81 49/81 64/81 81/81]
```

```
A =  
0.0400    0.1600    0.3600    0.6400    1.0000  
0.1111    0.2500    0.4444    0.9615    1.0000  
0.1837    0.3265    0.5102    0.7347    1.0000  
0.2500    0.3906    0.5625    0.7656    1.0000  
0.3086    0.4444    0.6049    0.7901    1.0000
```

The row-sum norm can be computed as 3.1481. Next, we can invert the matrix,



Question 4 (Q 10.15)

```
>> AI=inv(A)
```

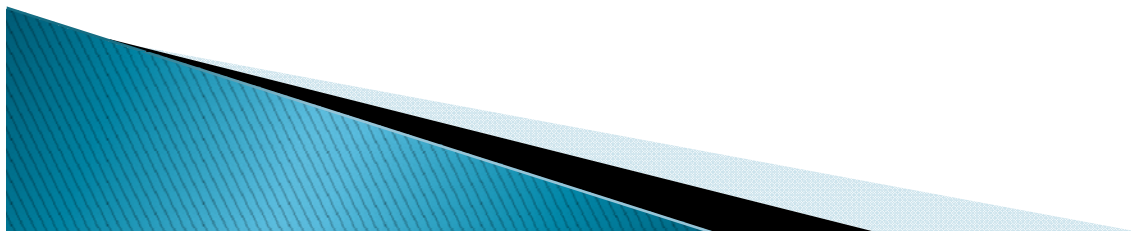
Warning: Matrix is close to singular or badly scaled.

Results may be inaccurate. RCOND = 2.230462e-018.

```
AI =
```

```
1.0e+016 *
```

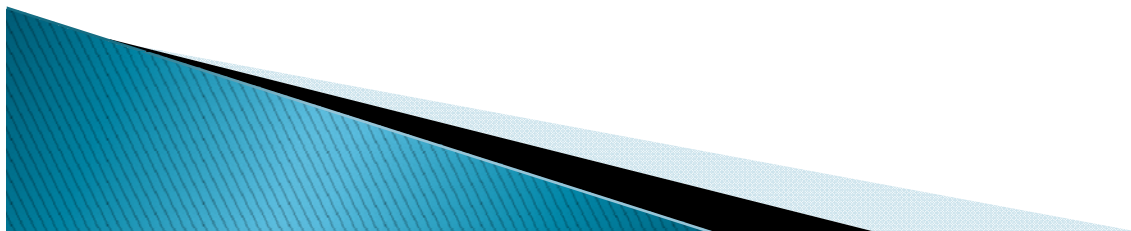
-0.0730	0	0.8581	-1.4945	0.7093
0.1946	-0.0000	-2.2884	3.9852	-1.8914
-0.1459	0	1.7163	-2.9889	1.4186
-0.0000	0.0000	-0.0000	-0.0000	0.0000
0.0243	-0.0000	-0.2860	0.4982	-0.2364



Question 4 (Q 10.15)

The row-sum norm of the inverse can be computed as 8.3596×10^{16} . The condition number can then be computed as

This corresponds to $\log_{10}(2.6317 \times 10^{17}) = 17.42$ suspect digits. Thus, as with (a), the suspect digits are more than the number of significant digits for the double precision representation used in MATLAB (15–16 digits). Consequently, we again can conclude that this matrix is highly ill-conditioned.



Thanks~

