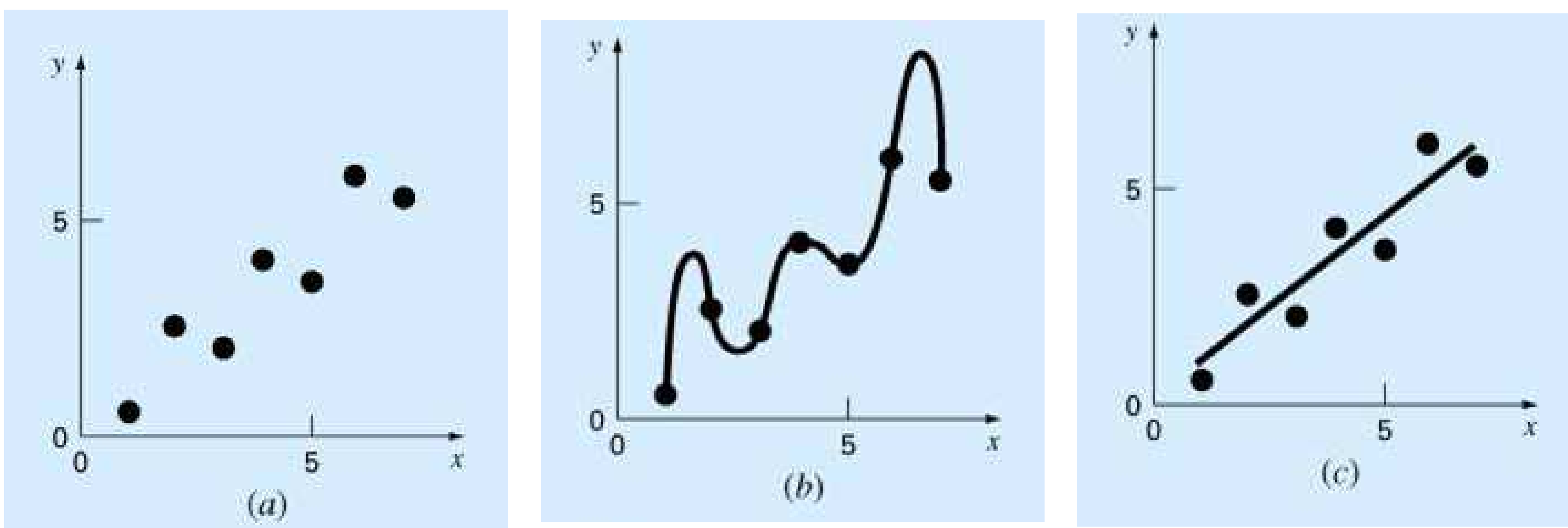
Chapter 6: Curve Fitting

Two types of curve fitting

- Least square regression
  Given data for discrete values, derive a single curve that represents the general trend of the data.
  — When the given data exhibit a significant degree of error or noise.

- Interpolation
  Given data for discrete values, fit a curve or a series of curves that pass directly through each of the points.
  — When data are very precise.



PART I: Least Square Regression

1 Simple Linear Regression

Fitting a straight line to a set of paired observations $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.
Mathematical expression for the straight line (model)

$$y = a_0 + a_1 x$$

where a_0 is the intercept, and a_1 is the slope.
Define

$$e_i = y_{i,measured} - y_{i,model} = y_i - (a_0 + a_1 x_i)$$

Criterion for a best fit:

$$\min S_r = \min_{a_0,a_1} \sum_{i=1}^{n} e_i^2 = \min_{a_0,a_1} \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2$$

Find a_0 and a_1:

$$\frac{\partial S_r}{\partial a_0} = -2\sum_{i=1}^{n}(y_i - a_0 - a_1 x_i) = 0 \quad (1)$$

$$\frac{\partial S_r}{\partial a_1} = -2\sum_{i=1}^{n}[(y_i - a_0 - a_1 x_i)x_i] = 0 \quad (2)$$

From (1), $\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} a_0 - \sum_{i=1}^{n} a_1 x_i = 0$, or

$$na_0 + \sum_{i=1}^{n} x_i a_1 = \sum_{i=1}^{n} y_i \quad (3)$$

From (2), $\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} a_0 x_i - \sum_{i=1}^{n} a_1 x_i^2 = 0$, or

$$\sum_{i=1}^{n} x_i a_0 + \sum_{i=1}^{n} x_i^2 a_1 = \sum_{i=1}^{n} x_i y_i \quad (4)$$

(3) and (4) are called normal equations.

From (3),

$$a_0 = \frac{1}{n}\sum_{i=1}^{n} y_i - \frac{1}{n}\sum_{i=1}^{n} x_i a_1 = \bar{y} - \bar{x} a_1$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$.

From (4), $\sum_{i=1}^n x_i(\frac{1}{n}\sum_{i=1}^n y_i - \frac{1}{n}\sum_{i=1}^n x_i a_1) + \sum_{i=1}^n x_i^2 a_1 = \sum_{i=1}^n x_i y_i$,

$$a_1 = \frac{\sum_{i=1}^n x_i y_i - \frac{1}{n}\sum_{i=1}^n x_i \sum_{i=1}^n y_i}{\sum_{i=1}^n x_i^2 - \frac{1}{n}\left(\sum_{i=1}^n x_i\right)^2}$$

or

$$a_1 = \frac{n\sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i}{n\sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i\right)^2}$$

Definitions:

$$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n}(y_i - a_0 - a_1 x_i)^2$$

Standard error of the estimate:

$$S_{y/x} = \sqrt{\frac{S_r}{n-2}}$$

— Spread around the regression line

Standard deviation of data points

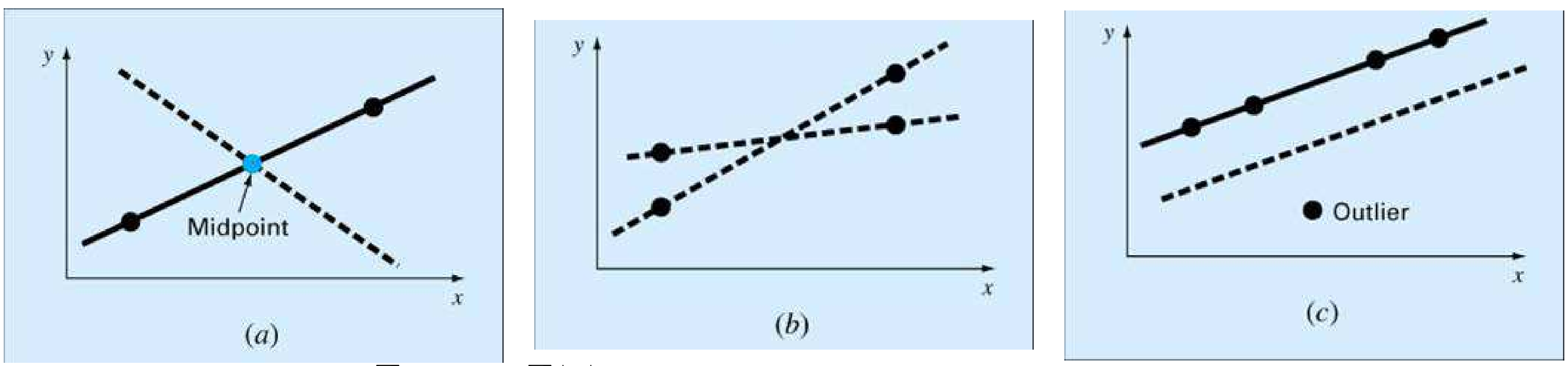
$$S_y = \sqrt{\frac{S_t}{n-1}} = \sqrt{\frac{\sum_{i=1}^n (y_i - \bar{y})^2}{n-1}}$$

where $S_t = \sum_{i=1}^{n}(y_i - \bar{y})^2$.
— Spread around the mean value $\bar{y}$.

Correlation coefficient:

$$r = \sqrt{\frac{S_t - S_r}{S_t}}$$

— Improvement or error reduction due to describing the data in terms of a straight line rather than as an average value.



Other criteria for regression (a) $\min \sum e_i$, (b) $\min \sum |e_i|$, and (c) $\min \max e_i$

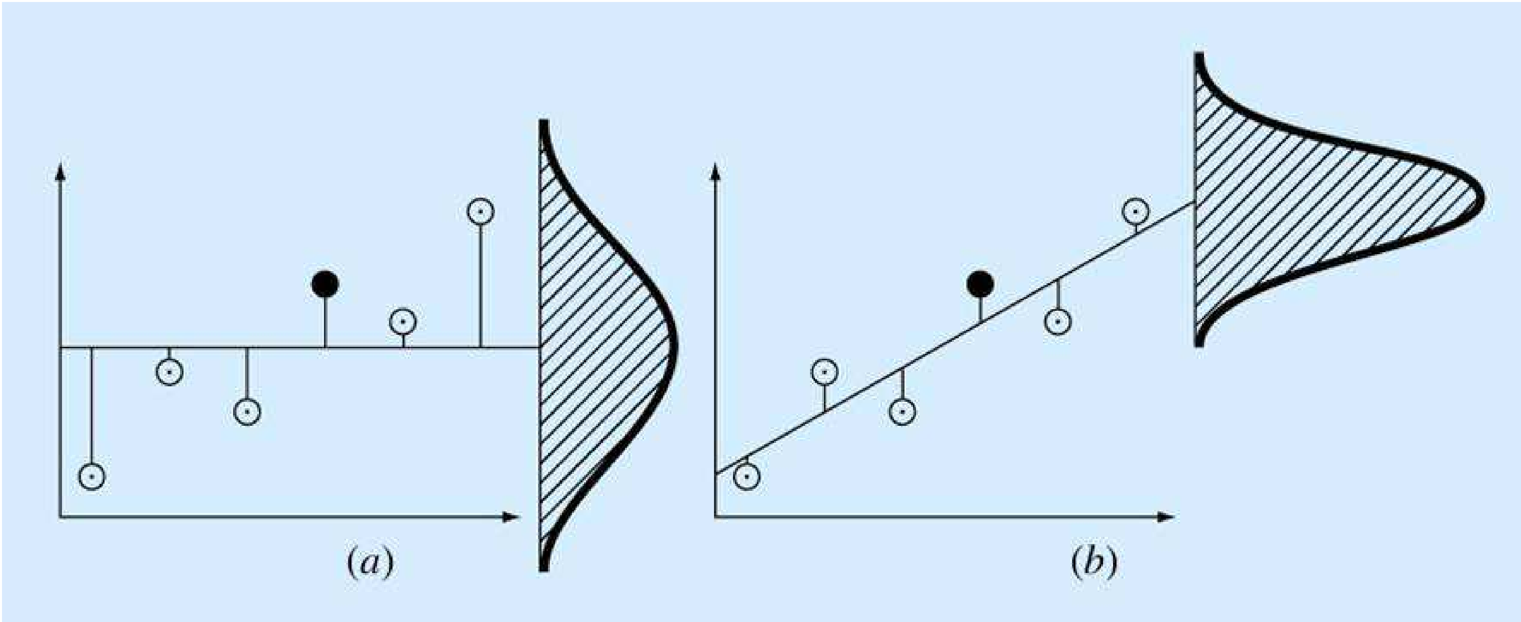


Figure 1: (a) Spread of data around mean of dependent variable, (b) spread of data around the best-fit line

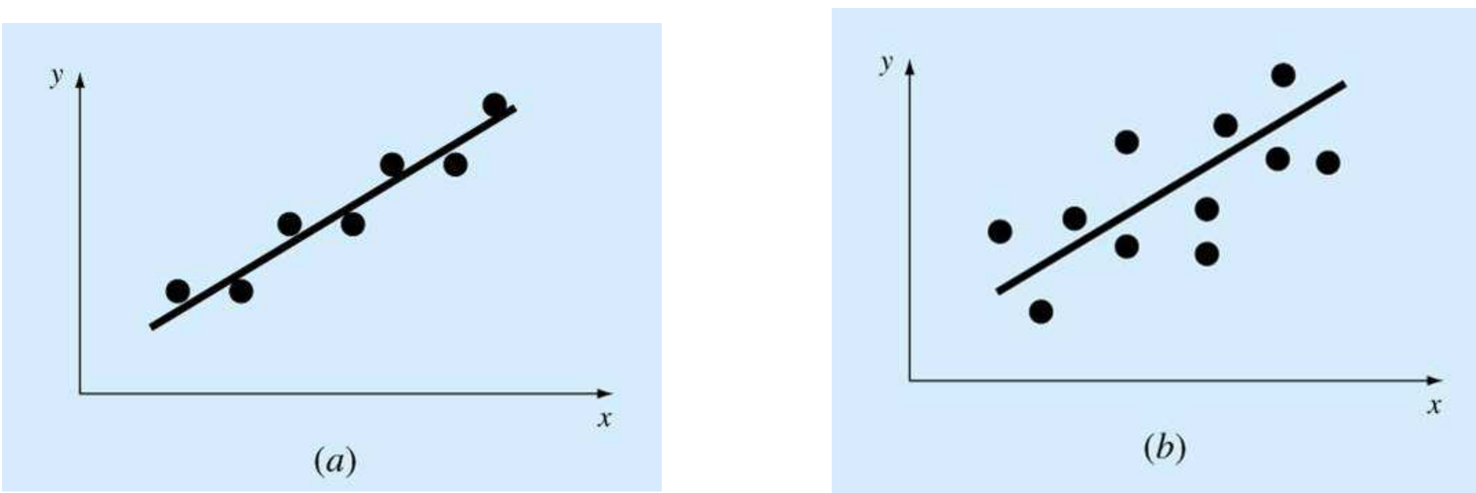


Illustration of linear regression with (a) small and (b) large residual errors

Example:

x	1	2	3	4	5	6	7
y	0.5	2.5	2.0	4.0	3.5	6.0	5.5

$\sum x_i = 1 + 2 + \ldots + 7 = 28$
$\sum y_i = 0.5 + 2.5 + \ldots + 5.5 = 24$
$\sum x_i^2 = 1^2 + 2^2 + \ldots + 7^2 = 140$
$\sum x_i y_i = 1 \times 0.5 + 2 \times 2.5 + \ldots + 7 \times 5.5 = 119.5$
$a_1 = \frac{n \sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i}{n \sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} = \frac{7 \times 119.5 - 28 \times 24}{7 \times 140 - 28^2} = 0.8393$
$a_0 = \bar{y} - \bar{x} a_1 = \frac{1}{n} \sum y_i - a_1 \frac{1}{n} \sum x_i = \frac{1}{7} \times 24 - 0.8393 \times \frac{1}{7} \times 28 = 0.07143.$
Model: $y = 0.07143 + 0.8393x.$

$S_r = \sum_{i=1}^n e_i^2,\ e_i = y_i - (a_0 + a_1 x_i)$
$e_1 = 0.5 - 0.07143 - 0.8393 \times 1 = -0.410$
$e_2 = 2.5 - 0.07143 - 0.8393 \times 2 = 0.750$
$e_3 = 2.0 - 0.07143 - 0.8393 \times 3 = -0.589$
$\ldots$
$e_7 = 5.5 - 0.07143 - 0.8393 \times 7 = -0.446$
$S_r = (-0.410)^2 + 0.750^2 + (-0.589)^2 + \ldots + 0.446^2 = 2.9911$

$S_t = \sum_{i=1}^{n}(y_i - \bar{y})^2 = 22.714$
Standard deviation of data points:
$S_y = \sqrt{\frac{S_t}{n-1}} = \sqrt{\frac{22.714}{6}} = 1.946$
Standard error of the estimate:
$S_{y/x} = \sqrt{\frac{S_r}{n-2}} = \sqrt{\frac{2.9911}{7-2}} = 0.774$
$S_{y/x} < S_y$, $S_r < S_t$.
Correlation coefficient $r = \sqrt{\frac{S_t - S_r}{S_t}} = \sqrt{\frac{22.714-2.9911}{22.714}} = 0.932$

2 Polynomial Regression

Given data (x_i, y_i), $i = 1, 2, \ldots, n$, fit a second order polynomial

$$y = a_0 + a_1 x + a_2 x^2$$

$e_i = y_{i,measured} - y_{i,model} = y_i - (a_0 + a_1 x_i + a_2 x_i^2)$
Criterion for a best fit:

$$\min S_r = \min_{a_0,a_1,a_2} \sum_{i=1}^{n} e_i^2 = \min_{a_0,a_1,a_2} \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i - a_2 x_i^2)^2$$

Find a_0, a_1, and a_2:

$$\frac{\partial S_r}{\partial a_0} = -2\sum_{i=1}^{n}(y_i - a_0 - a_1 x_i - a_2 x_i^2) = 0 \quad (1)$$

$$\frac{\partial S_r}{\partial a_1} = -2\sum_{i=1}^{n}[(y_i - a_0 - a_1 x_i - a_2 x_i^2)x_i] = 0 \quad (2)$$

$$\frac{\partial S_r}{\partial a_2} = -2\sum_{i=1}^{n}[(y_i - a_0 - a_1 x_i - a_2 x_i^2)x_i^2] = 0 \quad (3)$$

From (1), $\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} a_0 - \sum_{i=1}^{n} a_1 x_i - \sum_{i=1}^{n} a_2 x_i^2 = 0$, or

$$na_0 + \sum_{i=1}^{n} x_i a_1 + \sum_{i=1}^{n} x_i^2 a_2 = \sum_{i=1}^{n} y_i \quad (1^{'})$$

From (2), $\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} a_0 x_i - \sum_{i=1}^{n} a_1 x_i^2 - \sum_{i=1}^{n} x_i^3 a_2 = 0$, or

$$\sum_{i=1}^{n} x_i a_0 + \sum_{i=1}^{n} x_i^2 a_1 + \sum_{i=1}^{n} x_i^3 a_2 = \sum_{i=1}^{n} x_i y_i \quad (2^{'})$$

From (3), $\sum_{i=1}^{n} x_i^2 y_i - \sum_{i=1}^{n} a_0 x_i^2 - \sum_{i=1}^{n} a_1 x_i^3 - \sum_{i=1}^{n} x_i^4 a_2 = 0$, or

$$\sum_{i=1}^{n} x_i^2 a_0 + \sum_{i=1}^{n} x_i^3 a_1 + \sum_{i=1}^{n} x_i^4 a_2 = \sum_{i=1}^{n} x_i^2 y_i \quad (3^{'})$$

Comments:

- The problem of determining a least-squares second order polynomial is equivalent to solving a system of 3 simultaneous linear equations.
- In general, to fit an m-th order polynomial

  $$y = a_0 + a_1 x_1 + a_2 x^2 + \ldots + a_m x^m$$

  using least-square regression is equivalent to solving a system of $(m+1)$ simultaneous linear equations.
  Standard error: $S_{y/x} = \sqrt{\frac{S_r}{n-(m+1)}}$

3 Multiple Linear Regression

Multiple linear regression is used when y is a linear function of 2 or more independent variables.
Model: $y = a_0 + a_1 x_1 + a_2 x_2$.
Given data (x_{1i}, x_{2i}, y_i), $i = 1, 2, \ldots, n$
$e_i = y_{i,measured} - y_{i,model}$
$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_{1i} - a_2 x_{2i})^2$
Find a_0, a_1, and a_2 to minimize S_r.

$$\frac{\partial S_r}{\partial a_0} = -2\sum_{i=1}^{n}(y_i - a_0 - a_1x_{1i} - a_2x_{2i}) = 0 \quad (1)$$

$$\frac{\partial S_r}{\partial a_1} = -2\sum_{i=1}^{n}[(y_i - a_0 - a_1x_{1i} - a_2x_{2i})x_{1i}] = 0 \quad (2)$$

$$\frac{\partial S_r}{\partial a_2} = -2\sum_{i=1}^{n}[(y_i - a_0 - a_1x_{1i} - a_2x_{2i})x_{2i}] = 0 \quad (3)$$

From (1), $na_0 + \sum_{i=1}^{n} x_{1i}a_1 + \sum_{i=1}^{n} x_{2i}a_2 = \sum_{i=1}^{n} y_i \quad (1^{'})$

From (2), $\sum_{i=1}^{n} x_{1i}a_0 + \sum_{i=1}^{n} x_{1i}^2a_1 + \sum_{i=1}^{n} x_{1i}x_{2i}a_2 = \sum_{i=1}^{n} x_{1i}y_i \quad (2^{'})$

From (3), $\sum_{i=1}^{n} x_{2i}a_0 + \sum_{i=1}^{n} x_{1i}x_{2i}a_1 + \sum_{i=1}^{n} x_{2i}^2a_2 = \sum_{i=1}^{n} x_{2i}y_i \quad (3^{'})$

$$\begin{bmatrix} n & \sum x_{1i} & \sum x_{2i} \\ \sum x_{1i} & \sum x_{1i}^2 & \sum x_{1i}x_{2i} \\ \sum x_{2i} & \sum x_{1i}x_{2i} & \sum x_{2i}^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_{1i}y_i \\ \sum x_{2i}y_i \end{bmatrix}$$

Standard error: $S_{y/x} = \sqrt{\frac{S_r}{n-(m+1)}}$

4 General Linear Least Squares

Model:

$$y = a_0 Z_0 + a_1 Z_1 + a_2 Z_2 + \ldots + a_m Z_m$$

where $Z_0, Z_1, \ldots, Z_m$ are $(m+1)$ different functions.

Special cases:

- Simple linear LSR: $Z_0 = 1$, $Z_1 = x$, $Z_i = 0$ for $i \geq 2$
- Polynomial LSR: $Z_i = x^i$ ($Z_0 = 1$, $Z_1 = x$, $Z_2 = x^2$, ...)
- Multiple linear LSR: $Z_0 = 1$, $Z_i = x_i$ for $i \geq 1$

"Linear" indicates the model's dependence on its parameters, a_i's. The functions can be highly non-linear.

$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_{i,measured} - y_{i,model})^2$

Given data $(Z_{0i}, Z_{1i}, \ldots, Z_{mi}, y_i)$, $i = 1, 2, \ldots, n$,

$$S_r = \sum_{i=1}^{n} (y_i - \sum_{j=0}^{m} a_j Z_{ji})^2$$

Find a_j, $j = 0, 1, 2, \ldots, m$ to minimize S_r.

$$\frac{\partial S_r}{\partial a_k} = -2\sum_{i=1}^{n}(y_i - \sum_{j=0}^{m} a_j Z_{ji}) \cdot Z_{ki} = 0$$

$$\sum_{i=1}^{n} y_i Z_{ki} = \sum_{i=1}^{n}\sum_{j=0}^{m} Z_{ki} Z_{ji} a_j, \quad k = 0, 1, \ldots, m$$

$$\sum_{j=0}^{m}\sum_{i=1}^{n} Z_{ki} Z_{ji} a_j = \sum_{i=1}^{n} y_i Z_{ki}$$

$$Z^T Z A = Z^T Y$$

where

$$Z = \begin{bmatrix} Z_{01} & Z_{11} & \ldots & Z_{m1} \\ Z_{02} & Z_{12} & \ldots & Z_{m2} \\ & \ldots & & \\ Z_{0n} & Z_{1n} & \ldots & Z_{mn} \end{bmatrix}$$

PART II: Polynomial Interpolation

Given $(n+1)$ data points, (x_i, y_i), $i = 0, 1, 2, \ldots, n$, there is one and only one polynomial of order n that passes through all the points.

5 Newton's Divided-Difference Interpolating Polynomials

Linear Interpolation

Given (x_0, y_0) and (x_1, y_1)

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{f_1(x) - y_0}{x - x_0}$$

$$f_1(x) = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

$f_1(x)$: first order interpolation

A smaller interval, i.e., $|x_1 - x_0|$ closer to zero, leads to better approximation.

Example: Given $\ln 1 = 0$, $\ln 6 = 1.791759$, use linear interpolation to find $\ln 2$.

Solution:

$f_1(2) = \ln 2 = \ln 1 + \frac{\ln 6 - \ln 1}{6-1} \times (2-1) = 0.3583519$

True solution: $\ln 2 = 0.6931472.$

$\epsilon_t = |\frac{f_1(2) - \ln 2}{\ln 2}| \times 100\% = |\frac{0.3583519 - 0.6931472}{0.6931472}| \times 100\% = 48.3\%$

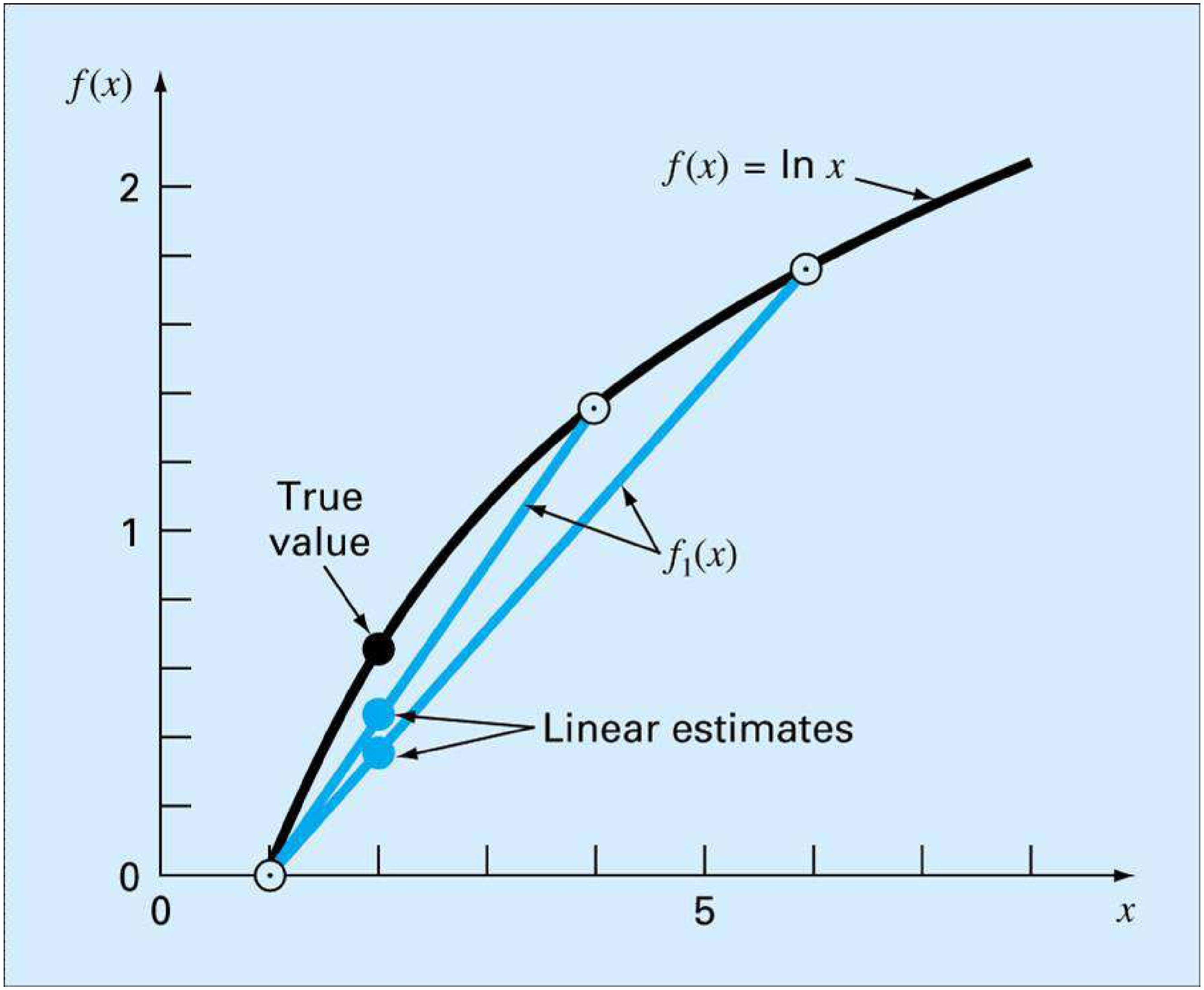


Figure 2: A smaller interval provides a better estimate

Quadratic Interpolation

Given 3 data points, (x_0, y_0), (x_1, y_1), and (x_2, y_2), we can have a second order polynomial

$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$
$f_2(x_0) = b_0 = y_0$
$f_2(x_1) = b_0 + b_1(x_1 - x_0) = y_1, \ \rightarrow \ b_1 = \frac{y_1 - y_0}{x_1 - x_0}$
$f_2(x_2) = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1) = y_2, \ \rightarrow \ b_2 = \frac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0}$ (*)

Proof (*):

$$\begin{aligned}
b_2 &= \frac{y_2 - b_0 - b_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} = \frac{y_2 - y_0 - \frac{(y_1 - y_0)(x_2 - x_0)}{x_1 - x_0}}{(x_2 - x_0)(x_2 - x_1)} \\
&= \frac{(y_2 - y_0)(x_1 - x_0) - (y_1 - y_0)(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)(x_1 - x_0)} \\
&= \frac{y_2(x_1 - x_0) - y_0 x_1 + y_0 x_0 - (y_1 - y_0)x_2 + y_1 x_0 - y_0 x_0}{(x_2 - x_0)(x_2 - x_1)(x_1 - x_0)} \\
&= \frac{y_2(x_1 - x_0) - y_1 x_1 + y_1 x_0 - (y_1 - y_0)x_2 + y_1 x_1 - y_0 x_1}{(x_2 - x_0)(x_2 - x_1)(x_1 - x_0)} \\
&= \frac{(y_2 - y_1)(x_1 - x_0) - (y_1 - y_0)(x_2 - x_1)}{(x_2 - x_0)(x_2 - x_1)(x_1 - x_0)}
\end{aligned}$$

Comments: In the expression of $f_2(x)$,

- $b_0 + b_1(x - x_0)$ is linear interpolating from (x_0, y_0) and (x_1, y_1), and
- $+b_2(x - x_0)(x - x_1)$ introduces second order curvature.

Example: Given $\ln 1 = 0$, $\ln 4 = 1.386294$, and $\ln 6 = 1.791759$, find $\ln 2$.

Solution:

$(x_0, y_0) = (1, 0)$, $(x_1, y_1) = (4, 1.386294)$, $(x_2, y_2) = (6, 1.791759)$

$b_0 = y_0 = 0$

$b_1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{1.386294 - 0}{4 - 1} = 0.4620981$

$b_2 = \frac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0} = \frac{\frac{1.791759 - 1.386294}{6 - 4} - 0.4620981}{6 - 1} = -0.0518731$

$f_2(x) = 0.4620981(x - 1) - 0.0518731(x - 1)(x - 4)$

$f_2(2) = 0.565844$

$\epsilon_t = |\frac{f_2(2) - \ln 2}{\ln 2}| \times 100\% = 18.4\%$

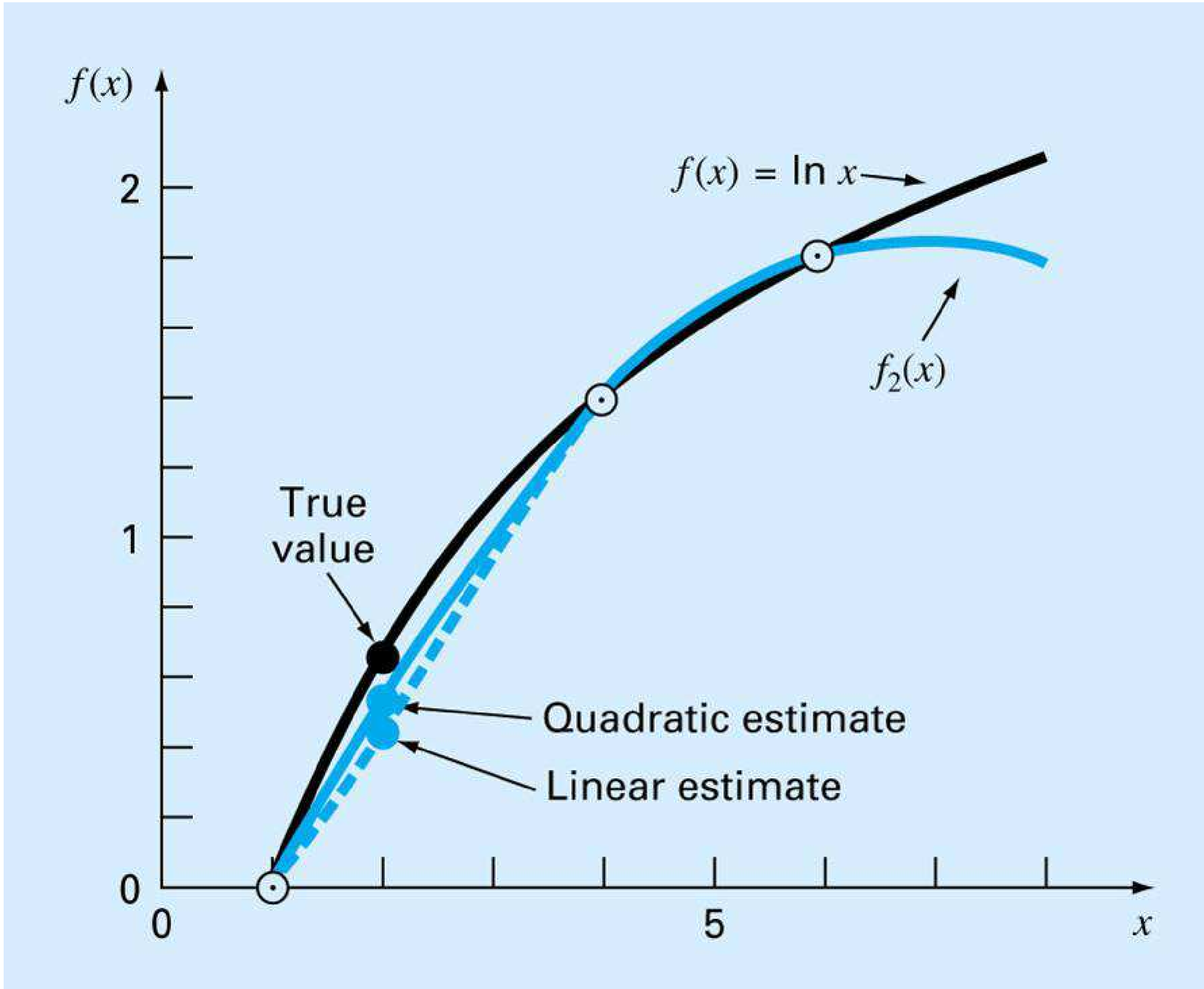


Figure 3: Quadratic interpolation provides a better estimate than linear interpolation

Straightforward Approach

$y = a_0 + a_1 x + a_2 x^2$

$$
\begin{aligned}
a_0 + a_1 x_0 + a_2 x_0^2 &= y_0 \\
a_0 + a_1 x_1 + a_2 x_1^2 &= y_1 \\
a_0 + a_1 x_2 + a_2 x_2^2 &= y_2
\end{aligned}
$$

or

$$
\begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}
=
\begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}
$$

General Form of Newton's Interpolating Polynomial

Given $(n+1)$ data points, (x_i, y_i), $i = 0, 1, \ldots, n$, fit an n-th order polynomial

$$f_n(x) = b_0 + b_1(x - x_0) + \ldots + b_n(x - x_0)(x - x_1) \ldots (x - x_n) = \sum_{i=0}^{n} b_i \prod_{j=0}^{i-1} (x - x_j)$$

find $b_0, b_1, \ldots, b_n$.

$x = x_0$, $y_0 = b_0$ or $b_0 = y_0$.

$x = x_1$, $y_1 = b_0 + b_1(x_1 - x_0)$, then $b_1 = \frac{y_1 - y_0}{x_1 - x_0}$

Define $b_1 = f[x_1, x_0] = \frac{y_1 - y_0}{x_1 - x_0}$.

$x = x_2$, $y_2 = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$, then $b_2 = \frac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0}$

Define $f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$, then $b_2 = f[x_2, x_1, x_0]$.

$\ldots$

$x = x_n$, $b_n = f[x_n, x_{n-1}, \ldots, x_1, x_0] = \frac{f[x_n, x_{n-1}, \ldots, x_1] - f[x_{n-1}, \ldots, x_1, x_0]}{x_n - x_0}$

6 Lagrange Interpolating Polynomials

The Lagrange interpolating polynomial is a reformulation of the Newton's interpolating polynomial that avoids the computation of divided differences. The basic format is

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where $L_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}$

<u>Linear Interpolation ($n = 1$)</u>

$f_1(x) = \sum_{i=0}^{1} L_i(x) f(x_i) = L_0(x) y_0 + L_1(x) y_1 = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$

$(f_1(x) = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0))$

<u>Second Order Interpolation ($n = 2$)</u>

$f_2(x) = \sum_{i=0}^{2} L_i(x) f(x_i) = L_0(x) y_0 + L_1(x) y_1 + L_2(x) y_2 = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$

Example: Given $\ln 1 = 0$, $\ln 4 = 1.386294$, and $\ln 6 = 1.791759$, find $\ln 2$.

Solution:

$(x_0, y_0) = (1, 0)$, $(x_1, y_1) = (4, 1.386294)$, $(x_2, y_2) = (6, 1.791759)$

$f_1(x) = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0) = \frac{x-4}{1-4} \times 0 + \frac{x-1}{4-1} \times 1.386294 = 0.4620981$

$f_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2 = \frac{(x-4)(x-6)}{(1-4)(1-6)} \times 0 + \frac{(x-1)(x-6)}{(4-1)(4-6)} \times 1.386294 + \frac{(x-1)(x-4)}{(6-1)(6-4)} \times 1.791760 = 0.565844$

Example: Find $f(2.6)$ by interpolating the following table of values.

i	x_i	y_i
1	1	2.7183
2	2	7.3891
3	3	20.0855

(1) Use Lagrange interpolation

$f_2(x) = \sum_{i=1}^{3} L_i(x) f(x_i)$, $L_i(x) = \prod_{j=1, j \neq i}^{3} \frac{x - x_j}{x_i - x_j}$

$$L_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} = \frac{(2.6-2)(2.6-3)}{(1-2)(1-3)} = -0.12$$

$$L_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} = \frac{(2.6-1)(2.6-3)}{(2-1)(2-3)} = 0.64$$

$$L_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} = \frac{(2.6-1)(2.6-2)}{(3-1)(3-2)} = 0.48$$

$$f_2(2.6) = -0.12 \times 2.7183 + 0.64 \times 7.3891 + 0.48 \times 20.08853 = 14.0439$$

(2) use Newton's interpolation

$$f_2(x) = b_0 + b_1(x - x_1) + b_2(x - x_1)(x - x_2)$$
$$b_0 = y_1 = 2.7183$$
$$b_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7.3891 - 2.7183}{2 - 1} = 4.6708$$
$$b_2 = \frac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0} = \frac{\frac{20.0855 - 7.3891}{3 - 2} - 4.6708}{3 - 1} = 4.0128$$
$$f_2(2.6) = 2.7183 + 4.6708 \times (2.6 - 1) + 4.0128 \times (2.6 - 1)(2.6 - 2) = 14.0439$$

(3) Use the straightforward method

$$f_2(x) = a_0 + a_1 x + a_2 x^2$$

$$a_0 + a_1 + a_2 \times 1^2 = 2.7183$$
$$a_0 + a_1 + a_2 \times 2^2 = 7.3891$$
$$a_0 + a_1 + a_2 \times 3^2 = 20.0855$$

or

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2.7183 \\ 7.3891 \\ 20.0855 \end{bmatrix}$$

$$[a_0 \; a_1 \; a_2]' = [6.0732; -7.3678 \; 4.0129]'$$

$$f(2.6) = 6.0732 - 7.3678 \times 2.6 + 4.01219 \times 2.6^2 = 14.044.$$

Example:

x_i	1	2	3	4
y_i	3.6	5.2	6.8	8.8

Model: $y = ax^b e^{cx}$

$\ln y = \ln a + b \ln x + cx$. Let $Y = \ln y$, $a_0 = \ln a$, $a_1 = b$, $x_1 = \ln x$, $a_2 = c$, and $x_2 = x$, then we have $Y = a_0 + a_1 x_1 + a_2 x_2$.

$x_{1,i}$	0	0.6931	1.0986	1.3863
$x_{2,i}$	1	2	3	4
Y_i	1.2809	1.6487	1.9169	2.1748

$\sum x_{1,i} = 3.1781$, $\sum x_{2,i} = 10$, $\sum x_{1,i}^2 = 3.6092$, $\sum x_{2,i}^2 = 30$, $\sum x_{1,i}x_{2,i} = 10.2273$, $\sum Y_i = 7.0213$, $\sum x_{1,i}Y_i = 6.2636$, $\sum x_{2,i}Y_i = 19.0280$. $n = 4$.

$$\begin{bmatrix} 1 & \sum x_{1,i} & \sum x_{2,i} \\ \sum x_{1,i} & \sum x_{1,i}^2 & \sum x_{2,i}x_{1,i} \\ \sum x_{2,i} & \sum x_{1,i}x_{2,i} & \sum x_{2,i}^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum Y_i \\ \sum x_{1,i}Y_i \\ \sum x_{2,i}Y_i \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3.1781 & 10 \\ 3.1781 & 3.6092 & 10.2273 \\ 10 & 10.2273 & 30 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 7.0213 \\ 6.2636 \\ 19.0280 \end{bmatrix}$$

$[a_0\ a_1\ a_2]' = [7.0213\ 6.2636\ 19.0280]'$

$a = e^{a_0} = 1.2332$, $b = a_1 = -1.4259$, $c = a_2 = 1.0505$, and

$y = ax^b e^{cx} = 1.2332 \cdot x^{-1.4259} \cdot e^{1.0505x}$.

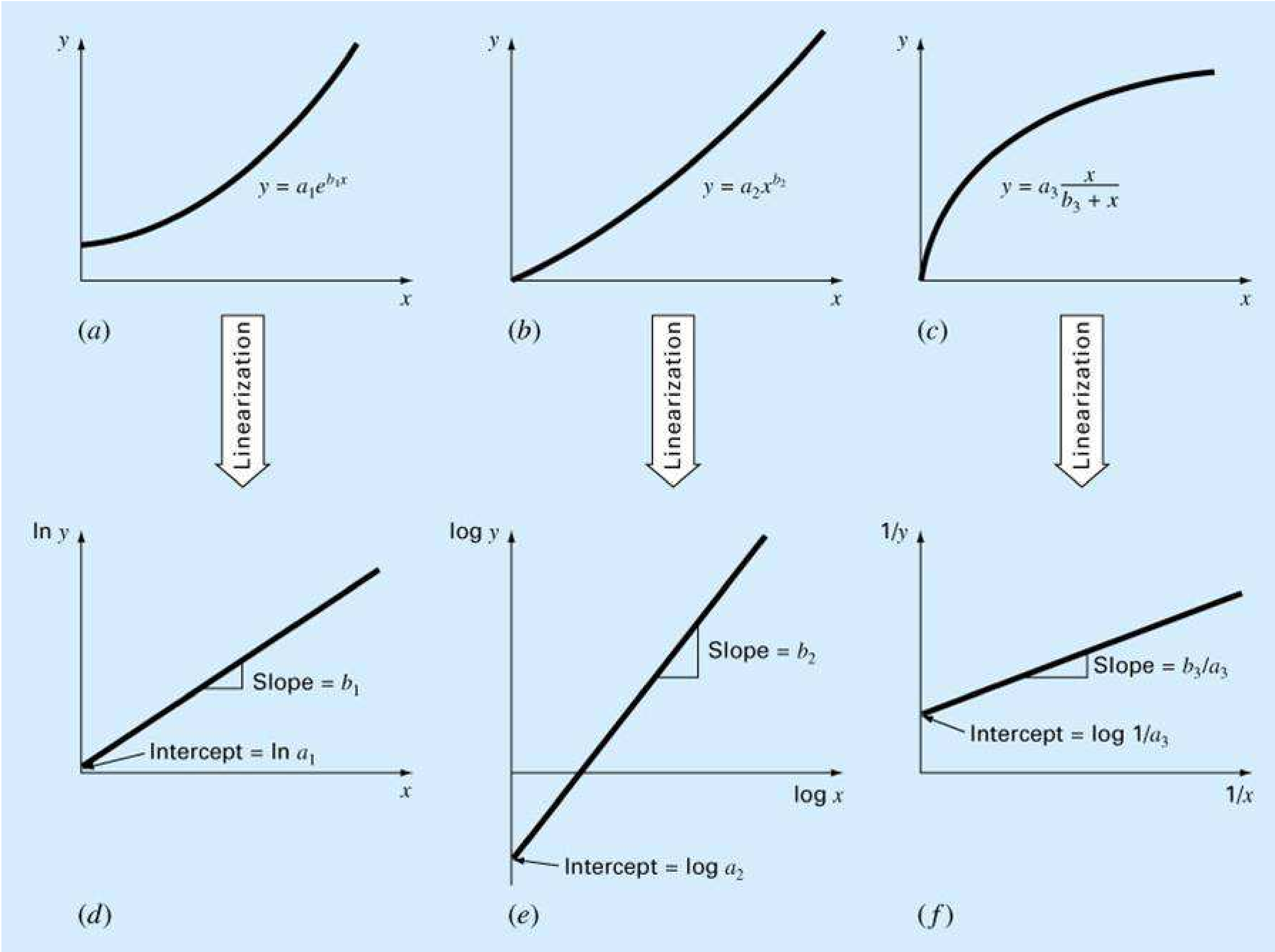


Figure 4: Linearization of nonlinear relationships