EE757
Numerical Techniques in Electromagnetics
Lecture 7

Recommended Text

A. Elsherbeni and V. Demir, The Finite Difference Time Domain Method for Electromagnetics with MATLAB Simulations, ACES Series on Computational Electromagnetics and Engineering, SciTech Publishing Inc. an Imprint of the IET, Second Edition, Edison, NJ, 2015

FDTD: an Introduction

the FDTD method is based on simple formulations that do not require complex Green's functions

it solves the problem in time, it can provide frequency-domain responses over a wide band using the Fourier transform

finite difference approximations are used to approximate time and space derivatives in Maxwell's differential equations

unlike TLM, the electric and magnetic fields are staggered in both time and space

Derivative Approximations

In 1966, Yee came up with a set of finite-difference equations for the time-dependent Maxwell's curl equations system

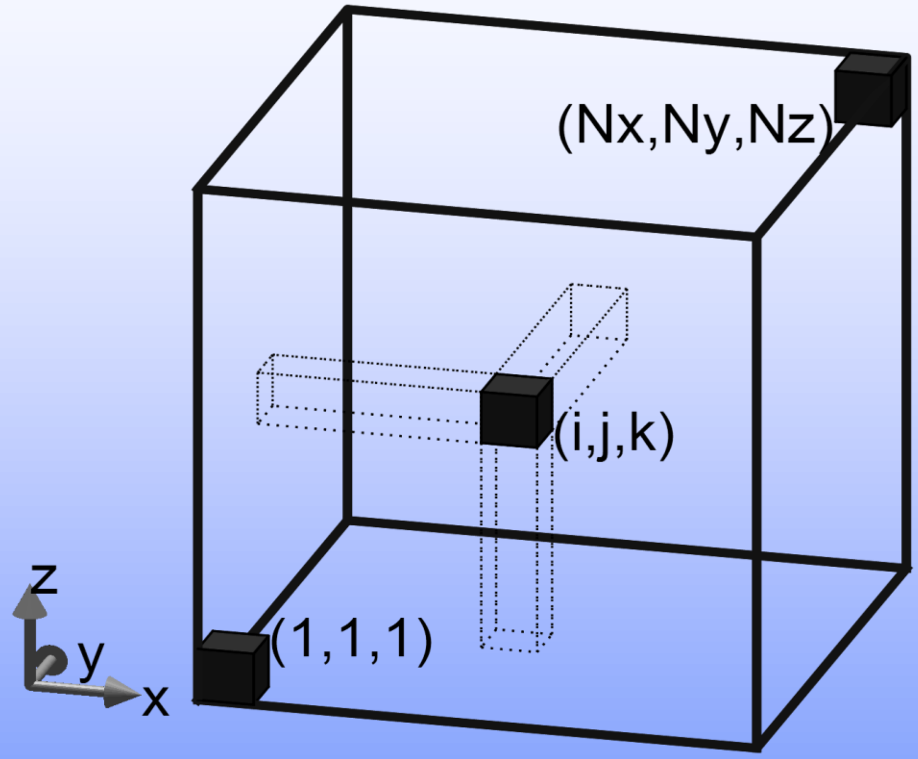
these equations employ the second-order accurate central difference formulas

for a function $f(x)$, its first-order derivative with respect to x is given by the second-order accurate formula:

$$f'(x) \approx \frac{f(x+\Delta x/2) - f(x-\Delta x/2)}{\Delta x}$$

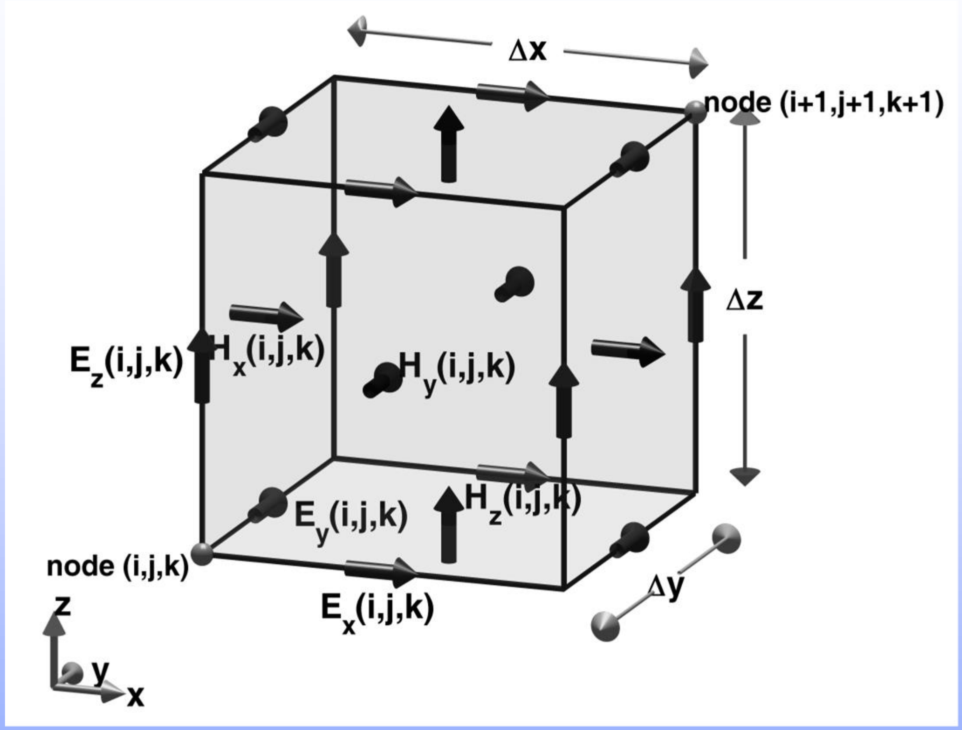
The same formula is applied to approximate time derivatives

The 3D FDTD Case



the domain is discretized into a grid of $(N_x \times N_y \times N_z)$ cells

Yee's Cell



electric field vector components are placed at the centers of the edges of the Yee's cell

magnetic field vector components are placed at the centers of the faces of the Yee's cell

Spatial Field Notation

$$E_x(i,j,k) \Rightarrow ((i-0.5)\Delta x,(j-1)\Delta y,(k-1)\Delta z),$$

$$E_y(i,j,k) \Rightarrow ((i-1)\Delta x,(j-0.5)\Delta y,(k-1)\Delta z),$$

$$E_z(i,j,k) \Rightarrow ((i-1)\Delta x,(j-1)\Delta y,(k-0.5)\Delta z),$$

$$H_x(i,j,k) \Rightarrow ((i-1)\Delta x,(j-0.5)\Delta y,(k-0.5)\Delta z),$$

$$H_y(i,j,k) \Rightarrow ((i-0.5)\Delta x,(j-1)\Delta y,(k-0.5)\Delta z),$$

$$H_z(i,j,k) \Rightarrow ((i-0.5)\Delta x,(j-0.5)\Delta y,(k-1)\Delta z).$$

Temporal Field Staggering

for a time-sampling period Δt, the electric field components are sampled at time instants $0, \Delta t, 2\Delta t, \ldots, n\Delta t, \ldots$

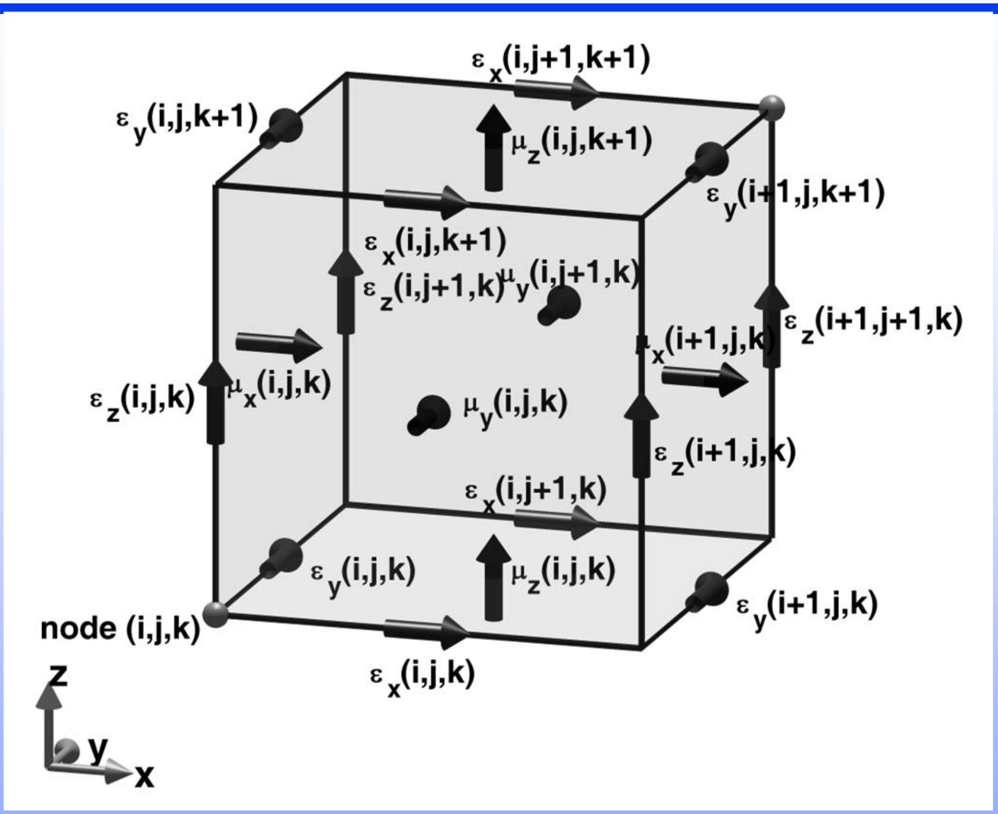
the magnetic field components are sampled at time instants

$$\frac{1}{2}\Delta t, (1+\frac{1}{2})\Delta t, \ldots, (n+\frac{1}{2})\Delta t, \ldots$$

Example: the *z*-component of the electric located at $((i-1)\Delta x, (j-1)\Delta y, (k-0.5)\Delta z)$ at time instant $n\Delta t$ is $E_z^n(i, j, k)$

the *y*-component of a magnetic field vector positioned at $((i-0.5)\Delta x, (j-1)\Delta y, (k-0.5)\Delta z)$ and sampled at time instant $(n+\frac{1}{2})\Delta t$ is $H_y^{n+\frac{1}{2}}(i, j, k)$

Material Interpolation



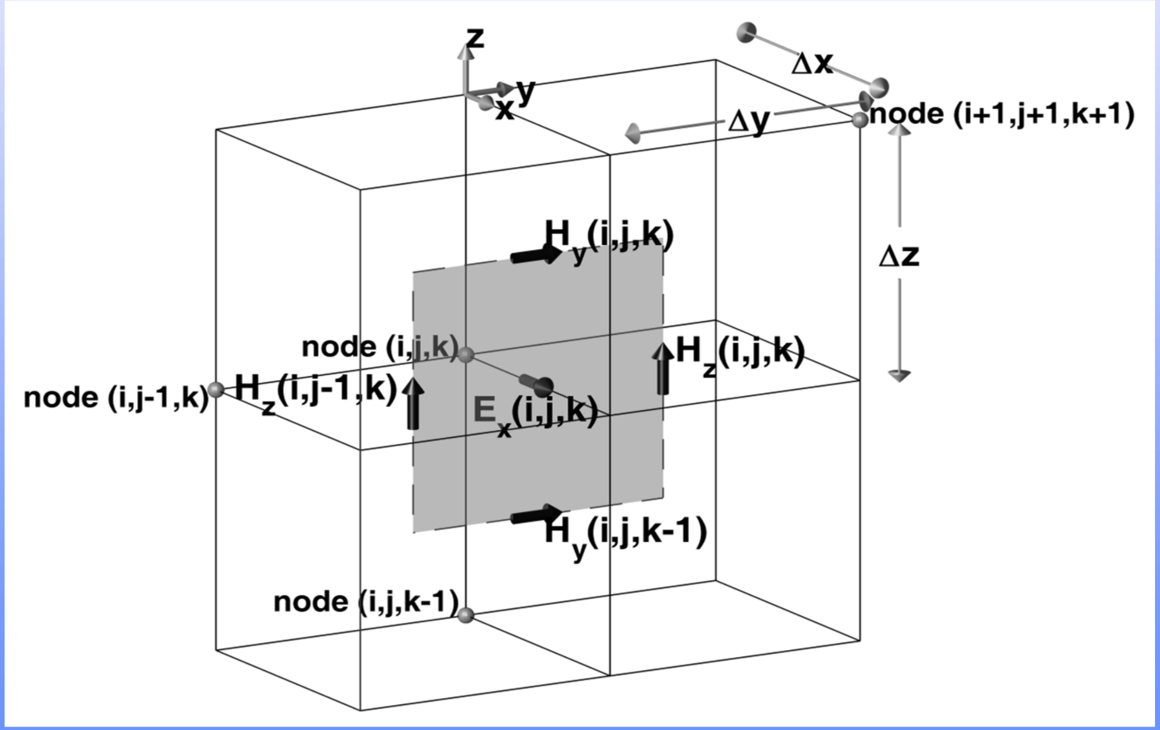
material properties are indexed in the same way as the field components

interpolation of material properties is a must. Why?

Discretization Example

consider the Maxwell's equation

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon_x}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \sigma_x^e E_x - J_{ix}\right),$$



Discretization Example (Cont'd)

$$\frac{E_x^{n+1}(i,j,k)-E_x^{n}(i,j,k)}{\Delta t}=+\frac{1}{\varepsilon_x(i,j,k)}\frac{H_z^{n+\frac{1}{2}}(i,j,k)-H_z^{n+\frac{1}{2}}(i,j-1,k)}{\Delta y}$$

$$-\frac{1}{\varepsilon_x(i,j,k)}\frac{H_y^{n+\frac{1}{2}}(i,j,k)-H_y^{n+\frac{1}{2}}(i,j,k-1)}{\Delta z}$$

$$-\frac{\sigma_x^e(i,j,k)}{\varepsilon_x(i,j,k)}E_x^{n+\frac{1}{2}}(i,j,k)-\frac{1}{\varepsilon_x(i,j,k)}J_{ix}^{n+\frac{1}{2}}(i,j,k).$$

the derivatives are approximated by central finite differences at the position of $E_x(i,j,k)$ and at the time instant $(n+0.5)\Delta t$

Discretization Example (Cont'd)

eliminating $E_x^{n+\frac{1}{2}}(i, j, k)$ we can write

$$E_x^{n+1}(i,j,k) = \frac{2\varepsilon_x(i,j,k) - \Delta t\sigma_x^e(i,j,k)}{2\varepsilon_x(i,j,k) + \Delta t\sigma_x^e(i,j,k)} E_x^n(i,j,k)$$

$$+ \frac{2\Delta t}{\left(2\varepsilon_x(i,j,k) + \Delta t\sigma_x^e(i,j,k)\right)\Delta y} \left(H_z^{n+\frac{1}{2}}(i,j,k) - H_z^{n+\frac{1}{2}}(i,j-1,k) \right)$$

$$- \frac{2\Delta t}{\left(2\varepsilon_x(i,j,k) + \Delta t\sigma_x^e(i,j,k)\right)\Delta z} \left(H_y^{n+\frac{1}{2}}(i,j,k) - H_y^{n+\frac{1}{2}}(i,j,k-1) \right)$$

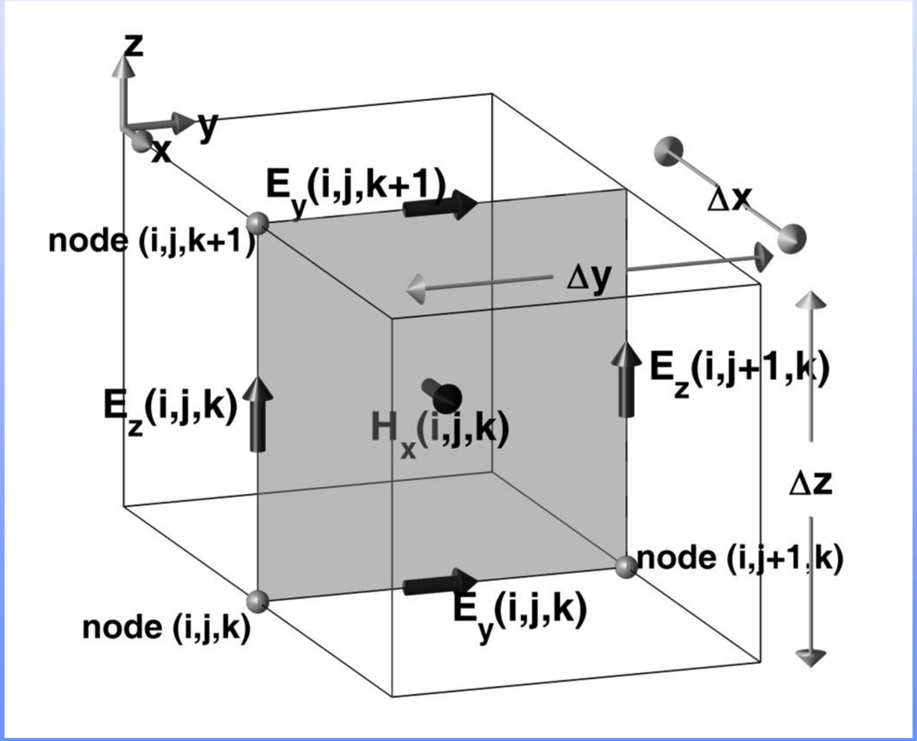
$$- \frac{2\Delta t}{2\varepsilon_x(i,j,k) + \Delta t\sigma_x^e(i,j,k)} J_{ix}^{n+\frac{1}{2}}(i,j,k).$$

electric field at time $(n+1)\Delta t$ is evaluated in terms of electric field at time instant $n\Delta t$ and magnetic fields at time instant $(n+0.5)\Delta t$

Second Discretization Example

consider the Maxwell's differential equation

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu_x}\left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}\right),$$



Second Discretization Example (Cont'd)

discretizing this equation at the location of $H_x(i,j,k)$ and at a time instant $n\Delta t$, we have

$$\frac{H_x^{n+\frac{1}{2}}(i,j,k) - H_x^{n-\frac{1}{2}}(i,j,k)}{\Delta t} =$$

$$\frac{1}{\mu_x(i,j,k)}\left(\frac{E_y^n(i,j,k+1) - E_y^n(i,j,k)}{\Delta z} - \frac{E_z^n(i,j+1,k) - E_z^n(i,j,k)}{\Delta y}\right)$$

reorganizing, we get the update equation

Second Discretization Example (Cont'd)

$$H_x^{n+\frac{1}{2}}(i,j,k) = H_x^{n-\frac{1}{2}}(i,j,k) + \frac{\Delta t}{\mu_x(i,j,k)}\left(\frac{E_y^n(i,j,k+1) - E_y^n(i,j,k)}{\Delta z} - \frac{E_z^n(i,j+1,k) - E_z^n(i,j,k)}{\Delta y}\right).$$

the magnetic field at time $(n+0.5)\Delta t$ is evaluated in terms of magnetic field at time instant $(n-0.5)\Delta t$ and the electric fields at time instant $n\Delta t$

The Electric Field Update Equations

$$E_x^{n+1}(i,j,k) = C_{exe}(i,j,k)\times E_x^{n}(i,j,k) + C_{exh}(i,j,k)\times\left(\frac{H_z^{n+\frac{1}{2}}(i,j,k)-H_z^{n+\frac{1}{2}}(i,j-1,k)}{\Delta y}+\frac{H_y^{n+\frac{1}{2}}(i,j,k)-H_y^{n+\frac{1}{2}}(i,j,k-1)}{\Delta z} - J_{ix}^{n+\frac{1}{2}}(i,j,k)\right),$$

$$E_y^{n+1}(i,j,k) = C_{eye}(i,j,k)\times E_y^{n}(i,j,k) + C_{eyh}(i,j,k)\times\left(\frac{H_x^{n+\frac{1}{2}}(i,j,k)-H_x^{n+\frac{1}{2}}(i,j,k-1)}{\Delta z}-\frac{H_z^{n+\frac{1}{2}}(i,j,k)-H_z^{n+\frac{1}{2}}(i-1,j,k)}{\Delta x} - J_{iy}^{n+\frac{1}{2}}(i,j,k)\right),$$

$$E_z^{n+1}(i,j,k) = C_{eze}(i,j,k)\times E_z^{n}(i,j,k) + C_{ezh}(i,j,k)\times\left(\frac{H_y^{n+\frac{1}{2}}(i,j,k)-H_y^{n+\frac{1}{2}}(i-1,j,k)}{\Delta x}-\frac{H_x^{n+\frac{1}{2}}(i,j,k)-H_x^{n+\frac{1}{2}}(i,j-1,k)}{\Delta y} - J_{iz}^{n+\frac{1}{2}}(i,j,k)\right),$$

The Electric Field Update Equations (Cont'd)

$$C_{eme}(i,j,k)=\frac{2\varepsilon_m(i,j,k)-\Delta t\sigma_m^e(i,j,k)}{2\varepsilon_m(i,j,k)+\Delta t\sigma_m^e(i,j,k)},$$

$$C_{emh}(i,j,k)=\frac{2\Delta t}{2\varepsilon_m(i,j,k)+\Delta t\sigma_m^e(i,j,k)},\ m=x,y,z.$$

update coefficients are calculated and stored before hand for all electric field components

Magnetic Field Update Equations

$$H_x^{n+\frac{1}{2}}(i,j,k) = H_x^{n-\frac{1}{2}}(i,j,k) \quad (2.24)$$

$$+ C_{hxe}(i,j,k) \times \left(\frac{E_y^n(i,j,k+1) - E_y^n(i,j,k)}{\Delta z} - \frac{E_z^n(i,j+1,k) - E_z^n(i,j,k)}{\Delta y} \right),$$

$$H_y^{n+\frac{1}{2}}(i,j,k) = H_y^{n-\frac{1}{2}}(i,j,k)$$

$$+ C_{hye}(i,j,k) \times \left(\frac{E_z^n(i+1,j,k) - E_z^n(i,j,k)}{\Delta x} - \frac{E_x^n(i,j,k+1) - E_x^n(i,j,k)}{\Delta z} \right),$$

$$H_z^{n+\frac{1}{2}}(i,j,k) = H_z^{n-\frac{1}{2}}(i,j,k)$$

$$+ C_{hze}(i,j,k) \times \left(\frac{E_x^n(i,j+1,k) - E_x^n(i,j,k)}{\Delta y} - \frac{E_y^n(i+1,j,k) - E_y^n(i,j,k)}{\Delta x} \right),$$

Magnetic Field Update Equations (Cont'd)

$$C_{hme}(i,j,k) = \frac{\Delta t}{\mu_m(i,j,k)}, m = x, y, z.$$

the magnetic field update coefficients are calculated beforehand

all coefficients are calculated using interpolated material properties for accurate results!

Algorithm Steps