EE750
Advanced Engineering Electromagnetics
Lecture 8

The Inhomogeneous Vector Potential Wave Equation

- Consider an infinitismal current source $\boldsymbol{J}_z$ placed at the origin

- For this configuration we have

$$\nabla^2 A_z + \beta^2 A_z = -\mu J_z \quad \text{(at source point)}$$

$$\nabla^2 A_z + \beta^2 A_z = 0 \quad \text{(at all other points)}$$

- We must have $A_z = A_z(r)$ because source is infinitismal
- It follows that at a general point we have

$$\nabla^2 A_z + \beta^2 A_z = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A_z}{\partial r}\right) + \beta^2 A_z = 0$$

$$\Downarrow$$

$$d^2 A_z / d r^2 + (2/r) d A_z / dr + \beta^2 A_z(r) = 0$$

The Inhomogenous Vector Equation (Cont'd)

- This equation has the general solution

$$A_z(r) = C_1 \frac{e^{-j\beta r}}{r} + C_2 \frac{e^{j\beta r}}{r}$$

Only the outward wave is a valid solution for our source

- For the corresponding static problem (β=0), we have

$$d^2 A_z / d\, r^2 + (2/r) d A_z / dr = 0 \Longrightarrow A_z(r) = \frac{C_1}{r}$$

- It follows that the solutions of time-harmonic problems can be obtained from the corresponding static solutions by multiplying by the phase retardation factor exp($-j\beta r$) for each source point

The Inhomogenous Vector Equation (Cont'd)

- Using similarity with the solution of the static charge equation we obtain for an arbitrary current distribution

$$A_z = \frac{\mu}{4\pi} \iiint_{V'} \frac{J_z}{r} dV'$$

- For the time-harmonic case, we have

$$A_z = \frac{\mu}{4\pi} \iiint_{V'} \frac{J_z}{r} e^{-j\beta r} dV'$$

- For sources with different orientation, we have

$$\nabla^2 A_x + \beta^2 A_x = -\mu J_x \Rightarrow A_x = \frac{\mu}{4\pi} \iiint_{V'} \frac{J_x}{r} e^{-j\beta r} dV'$$

$$\nabla^2 A_y + \beta^2 A_y = -\mu J_y \Rightarrow A_y = \frac{\mu}{4\pi} \iiint_{V'} \frac{J_y}{r} e^{-j\beta r} dV'$$

The Inhomogenous Vector Equation (Cont'd)

- In general, we have $\boldsymbol{A} = \dfrac{\mu}{4\pi} \iiint_{V'} \dfrac{\boldsymbol{J}}{r} e^{-j\beta r}\, dV'$

- It follows that for a general source and observation point

$$\boldsymbol{A}(x, y, z) = \frac{\mu}{4\pi} \iiint_{V'} \frac{\boldsymbol{J}(x', y', z')}{R} e^{-j\beta R}\, dV'$$

$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

- Similarly, the electric vector potential may be expressed as

$$\boldsymbol{F}(x, y, z) = \frac{\varepsilon}{4\pi} \iiint_{V'} \frac{\boldsymbol{M}(x', y', z')}{R} e^{-j\beta R}\, dV'$$

The Inhomogenous Vector Equation (Cont'd)

- Notice that the volume integral reduces to a surface or line integrals if we have surface or linear current densities, respectively

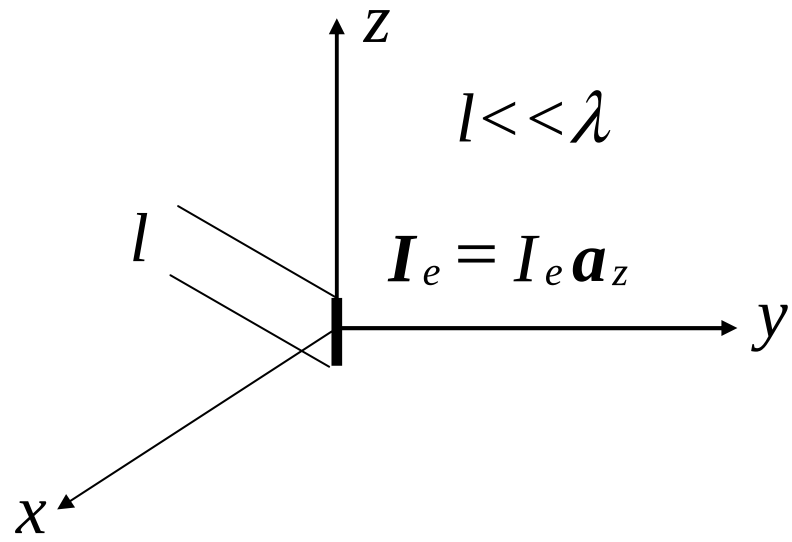
$$\boldsymbol{A}(x,y,z)=\frac{\mu}{4\pi}\iint_{S'}\frac{\boldsymbol{J}_s(x',y',z')}{R}e^{-j\beta R}\,dS'$$

$$\boldsymbol{F}(x,y,z)=\frac{\varepsilon}{4\pi}\iint_{S'}\frac{\boldsymbol{M}_s(x',y',z')}{R}e^{-j\beta R}\,dS'$$

$$\boldsymbol{A}(x,y,z)=\frac{\mu}{4\pi}\int_{C}\frac{\boldsymbol{I}_e(x',y',z')}{R}e^{-j\beta R}\,d\ell'$$

$$\boldsymbol{F}(x,y,z)=\frac{\varepsilon}{4\pi}\int_{C}\frac{\boldsymbol{I}_m(x',y',z')}{R}e^{-j\beta R}\,d\ell'$$

Example:An Infinitismal Dipole



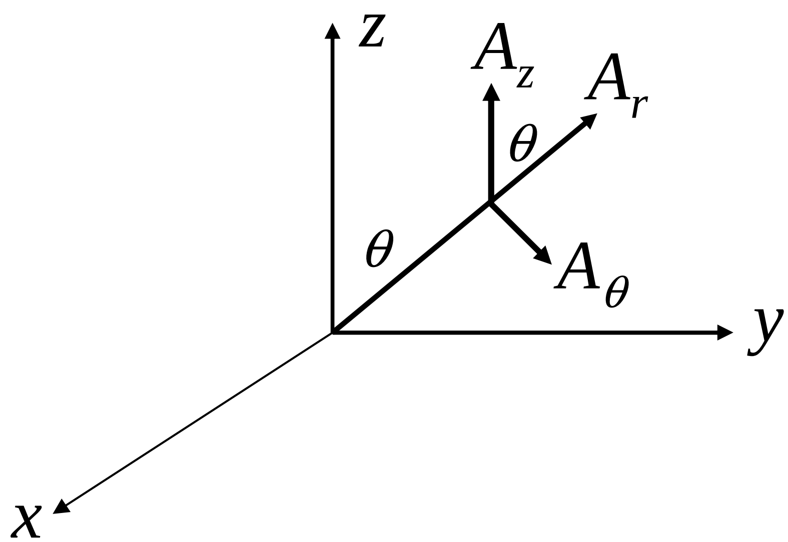
- We start by obtaining the magnetic vector potential

$$A_z(x, y, z) = \frac{\mu}{4\pi} \int_{-l/2}^{l/2} \frac{\boldsymbol{a}_z I_e(x', y', z')}{R} e^{-j\beta R} dz'$$

$$\Downarrow$$

$$A_z(x, y, z) = \frac{\boldsymbol{a}_z \mu I_e}{4\pi r} e^{-j\beta r} \int_{-l/2}^{l/2} dz' = \frac{\boldsymbol{a}_z \mu I_e \, l}{4\pi r} e^{-j\beta r}$$

Example:An Infinitismal Dipole (Cont'd)



- For spherical wave we usually utilize spherical coordinates

$$A_r = A_z\, cos\theta = \frac{\mu I_e\, l}{4\pi r} e^{-j\beta r} cos\theta$$

$$A_\theta = \text{-} A_z\, sin\theta = \frac{-\mu I_e\, l}{4\pi r} e^{-j\beta r} sin\theta$$

$$A_\varphi = 0$$

Example:An Infinitismal Dipole (Cont'd)

- We then proceed to obtain the magnetic and electric fields

$$\boldsymbol{H} = \frac{1}{\mu}\nabla\times\boldsymbol{A} \quad \Longrightarrow \quad \boldsymbol{H} = \boldsymbol{a}_{\varphi}\frac{1}{\mu r}\left(\frac{\partial}{\partial r}(r\,A_{\theta}) - \frac{\partial A_r}{\partial\theta}\right)$$

$$H_{\varphi} = \frac{j\beta I_e\,l}{4\pi r}\sin\theta\left(1+\frac{1}{j\beta r}\right)e^{-j\beta r}\;,\; H_r = H_{\theta} = 0$$

$$\boldsymbol{E} = -j\omega\boldsymbol{A} - \frac{j}{\omega\mu\varepsilon}\nabla(\nabla.\boldsymbol{A})$$

$$E_r = \frac{\eta I_e\,l}{2\pi\,r^2}\cos\theta\left(1+\frac{1}{j\beta r}\right)e^{-j\beta r}$$

$$E_{\theta} = \frac{j\eta\beta I_e\,l}{4\pi r}\sin\theta\left(1+\frac{1}{j\beta r}-\frac{1}{(\beta r)^2}\right)e^{-j\beta r}\;,\; E_{\varphi} = 0$$

Example: An Infinitismal Dipole (Cont'd)

- For far fields we keep only the $(1/r)$ terms. It follows that we have

$$H_\varphi \approx \frac{j\beta I_e\, l}{4\pi r} \sin\theta\, e^{-j\beta r}$$

$$E_r \approx 0$$

$$E_\theta \approx \frac{j\eta\beta I_e\, l}{4\pi r} \sin\theta\, e^{-j\beta r}$$

$$E_\theta / H_\varphi = \eta$$

One-Dimensional Finite Differences (FD)

- Our target is to solve a given differential equation
- The computational domain is filled with a mesh and we care only about the values of the unknown function at these grid points $x_i=ih$, $f_i=f(ih)$
- Derivatives are then approximated by their finite-differences

$$f'(x)=\frac{f(x+h)-f(x)}{h}+o(h)$$ forward difference

$$f'(x)=\frac{f(x)-f(x-h)}{h}+o(h)$$ backward difference

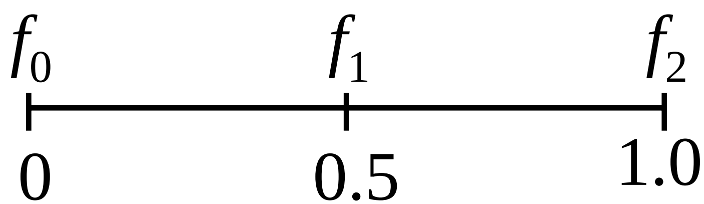
$$f'(x)=\frac{f(x+h)-f(x-h)}{2h}+o(h^2)$$ central difference

$$f''(x)=\frac{f(x+h)-2f(x)+f(x-h)}{h^2}+o(h^2)$$

One-Dimensional Example

- Solve $d^2f/dx^2+4f=0$, for $0 \le x \le 1.0$

Subject to the boundary conditions $f(0)=0$, $f(1)=1$

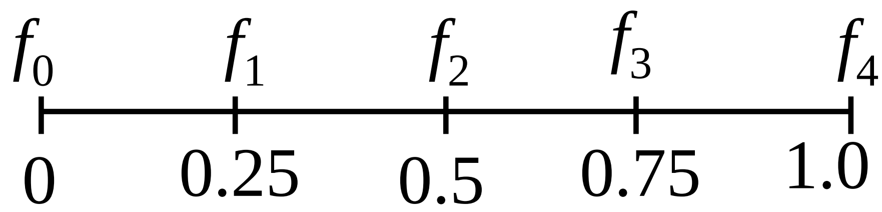


For the shown discretization, we have only one unknown and we need one equation

$$\frac{f_2 - 2f_1 + f_0}{0.5^2} + 4f_1 = 0 \quad \Rightarrow \quad f_1 = f_0 + f_2$$

$f_1=1$

One-Dimensional Example (Cont'd)



For the shown finer discretization, we have only three unknowns and we need three equations

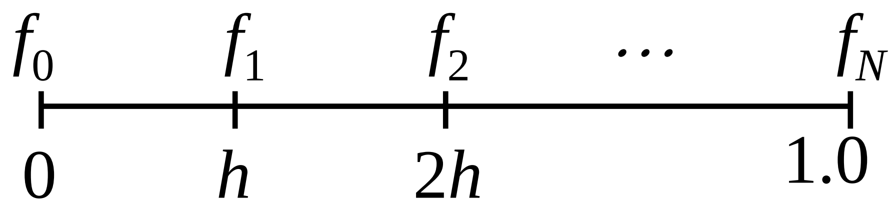
$$\frac{f_2 - 2f_1 + f_0}{0.25^2} + 4f_1 = 0$$

$$\frac{f_3 - 2f_2 + f_1}{0.25^2} + 4f_2 = 0$$

$$\frac{f_4 - 2f_3 + f_2}{0.25^2} + 4f_3 = 0$$

three equations in three unknowns

One-Dimensional Example (Cont'd)



$h=1/N$

- For the shown fine discretization, we have only (N-1) unknowns and we need (N-1) equations

- The equation of the ith node is

$$\frac{f_{i+1}-2f_i+f_{i-1}}{h^2}+4f_i=0$$

- We then solve the (N-1) equations in the (N-1) unknowns

Two-Dimensional FD

- The computational domain is filled with a two-dimensional mesh and we care only about the values of the unknown function at these grid points $x_i=ih$, $y_j=jh$ $f_{i,j}=f(ih, jh)$
- Derivatives are then approximated by finite-differences

$$\frac{df}{dx} \approx \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{df}{dx} \approx \frac{f(x,y) - f(x-h,y)}{h}$$

$$\frac{df}{dx} \approx \frac{f(x+h,y) - f(x-h,y)}{2h}$$

$$\frac{d^2 f}{d x^2} \approx \frac{f(x+h,y) - 2f(x,y) + f(x-h,y)}{h^2}$$

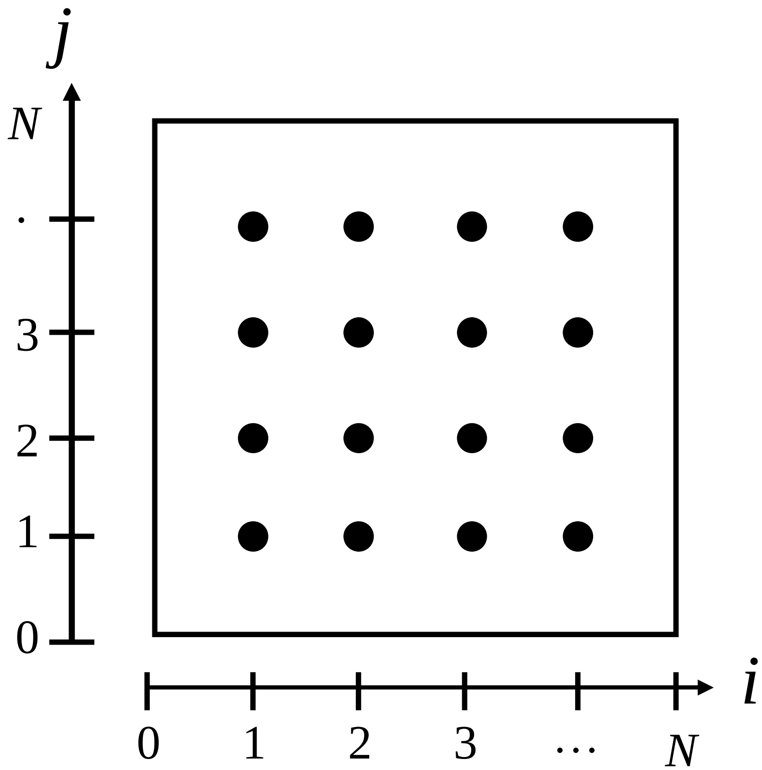
$$\frac{df}{dy} \approx \frac{f(x,y+h) - f(x,y)}{h}$$

$$\frac{df}{dy} \approx \frac{f(x,y) - f(x,y-h)}{h}$$

$$\frac{df}{dy} \approx \frac{f(x,y+h) - f(x,y+h)}{2h}$$

$$\frac{d^2 f}{d y^2} \approx \frac{f(x,y+h) - 2f(x,y) + f(x,y-h)}{h^2}$$

Two-Dimensional FD Example



- Solve the Laplace equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

subject to mixed Dirichlet and Neumann Boundary conditions

Two-Dimensional FD Example (Cont'd)

- At an arbitrary point (*ih*,*jh*), the FD scheme is

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\approx\frac{f(x+h,y)-2f(x,y)+f(x-h,y)}{h^2}$$

$$+\frac{f(x,y+h)-2f(x,y)+f(x,y-h)}{h^2}=0$$

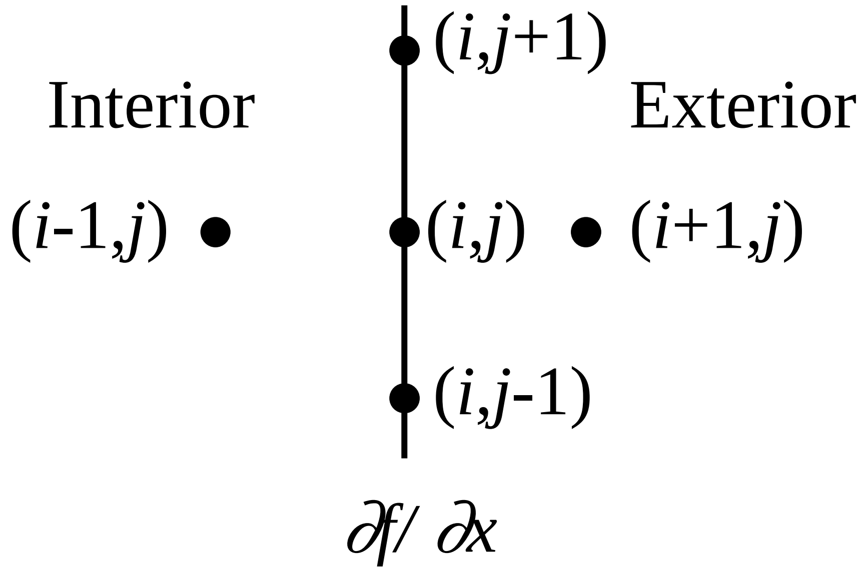
$$\Downarrow$$

$$\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)_{ih,jh}\approx\frac{f_{i+1,j}+f_{i-1,j}+f_{i,j+1}+f_{i,j-1}-4f_{i,j}}{h^2}=0$$

- This FD scheme works fine for all interior points and Dirichlet boundary points

- Neumann boundary points add more unknowns

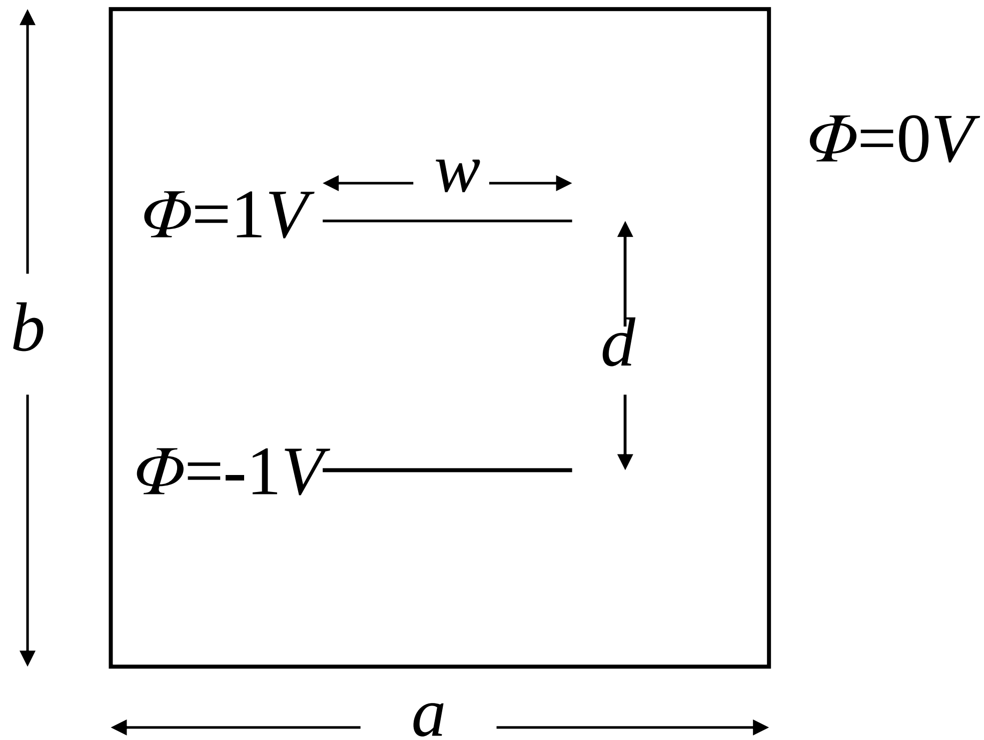
Two-Dimensional FD Example (Cont'd)



- At any Neumann boundary point (i,j), we add a fictitious external node satisfying $f_{i+1,j}=f_{i-1,j}$
- An equation is then added for each Neumann point

$$\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)_{ih,\,jh} \approx \frac{2f_{i-1,j}+f_{i,j+1}+f_{i,j-1}-4f_{i,j}}{h^2}=0$$

Application



- Obtain the electric potential everywhere