EE2CI5 Lab 4: RLC Circuits

1 Objective

The objective of this laboratory is to investigate the behaviour of underdamped and critically-damped RLC circuits.

2 Equipment

Function generator
Variable power supply
Multi-meter
Oscilloscope
Hook-up wire
Resistors: $100\Omega, 200\Omega, 470\Omega,\ 1\text{k}\Omega, 2\text{k}\Omega, 2.2\text{k}\Omega, 2.4\text{k}\Omega, 4.7\text{k}\Omega$
Capacitors: 0.1µF, 1.0µF
Large inductor: 850mH.

Note: We may not have 200Ω and $2.4\text{k}\Omega$ resistors available. If not, then use a combination of the resistors that you have, arranged in series or parallel.

3 Theoretical Analysis

Given a second-order differential equation of the form

$$\frac{d^2}{dt^2}x(t) + a_1\frac{d}{dt}x(t) + a_0x(t) = f(t) \tag{1}$$

we can define the undamped natural frequency and the damping ratio as

$$\omega_0 = \sqrt{a_0} \qquad \text{and} \qquad \zeta = \frac{a_1}{2\omega_0}, \tag{2}$$

respectively. The damping ratio is sometimes called the damping factor.

Now, consider the characteristic equation of (1),

$$s^2 + a_1s + a_0 = 0. \tag{3}$$

Using the notions of undamped natural frequency and damping ratio, we can write the solutions of this equation as

$$s_1, s_2 = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}. \tag{4}$$

From this equation it can be seen that ζ plays a significant role in the nature of the solutions.

- If $\zeta > 1$:
  - there are two distinct real solutions to (3), s_1, s_2;
  - the natural response of the differential equation in (1) takes the form

$$x_n(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}; \tag{5}$$

  - and the system described by the differential equation is said to be over-damped.
- If $\zeta = 1$:
  - there are two identical real solutions to (3), $s_1, s_2 = -\zeta\omega_0 = -\sigma$;
  - the natural response of the differential equation in (1) takes the form

$$x_n(t) = A_1 e^{-\sigma t} + A_2 t e^{-\sigma t}; \tag{6}$$

  - and the system described by the differential equation is said to be critically-damped.
- If $\zeta < 1$:
  - there are two complex valued solutions to (3) that form a complex-conjugate pair, $s_1, s_2 = -\zeta\omega_0 \pm j\omega_0\sqrt{1-\zeta^2} = -\sigma \pm j\omega_d$, where ω_d is sometimes called the damped natural frequency.
  - the natural response of the differential equation in (1) takes the form

$$\begin{aligned} x_n(t) &= A_1 e^{-\sigma t}\cos(\omega_d t) + A_2 e^{-\sigma t}\sin(\omega_d t) && \text{(7a)} \\ &= B e^{-\sigma t}\cos(\omega_d t - \phi), && \text{(7b)} \end{aligned}$$

    where $B = \sqrt{A_1^2 + A_2^2}$ and, when $A_1 > 0$, $\phi = \mathrm{atan}(A_2/A_1)$;
  - and the system described by the differential equation is said to be under-damped.

4 Pre-Lab Exercise (3 marks)

Derive expressions for the natural frequency and the damping ratio for a parallel combination of a resistor of R Ohms, an inductor of L Henries, and a capacitor of C Farads. To do so, you first need to write a differential equation that describes the operation of such a circuit. Then you can apply equations (1) and (2). Your derivation should result in

$$\zeta_{\text{parallel RLC}} = \frac{1}{2R}\sqrt{\frac{L}{C}}. \tag{8}$$

Repeat the above analysis for a series RLC circuit to show that

$$\zeta_{\text{series RLC}} = \frac{R}{2}\sqrt{\frac{C}{L}}. \tag{9}$$

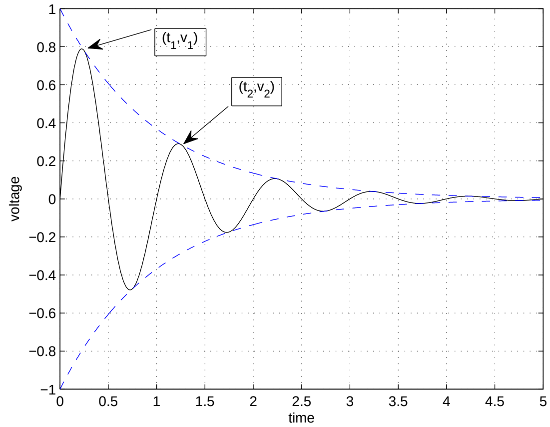

Figure 1: A voltage in an under-damped circuit.

5 Experiments

5.1 Under-damped circuits (2 marks)

In this experiment you will construct an under-damped series RLC circuit and then measure ω_0 and ζ.

1. Based on theoretical calculations, choose appropriate values for R, L and C to construct an under-damped series RLC circuit. For reasons discussed below, I suggest that you try to choose these parameters in such a way that ζ is just below 0.2.

2. Select a circuit scenario in which you will be able to measure the natural response.

3. Now select a setting for the power supply that will mimic that scenario so that you can view the natural response of the circuit on the oscilloscope.

4. Use measurements of the natural response to estimate the undamped natural frequency, ω_0, and the damping ratio, ζ. The discussion below provides a reasonable technique for doing so when ζ is small.

Consider the sketch in Figure 1 of a voltage in a certain under-damped circuit. The difference between t_2 and t_1 is one period corresponding to the damped natural frequency, $\omega_d = \omega_0\sqrt{1-\zeta^2}$. However, if ζ is much smaller than 1, this is quite close to ω_0. Indeed, you can choose ζ so that the error incurred in the approximation that ω_0 is equal to ω_d is likely to be less than the errors in your measurements from the oscilloscope. If you feel confident that your circuit has a small ζ, then you can estimate ω_0 as $2\pi/(t_2-t_1)$.

To complete our experiment we will need another measurement. Note that the marked points on the graph both represent times when the cosine term in (7b) is equal to 1. Therefore,

$$\frac{v_2}{v_1} = e^{-\sigma(t_2-t_1)} = e^{-\omega_0\zeta(t_2-t_1)} = e^{-2\pi\zeta}. \tag{10}$$

This can be used to estimate ζ.

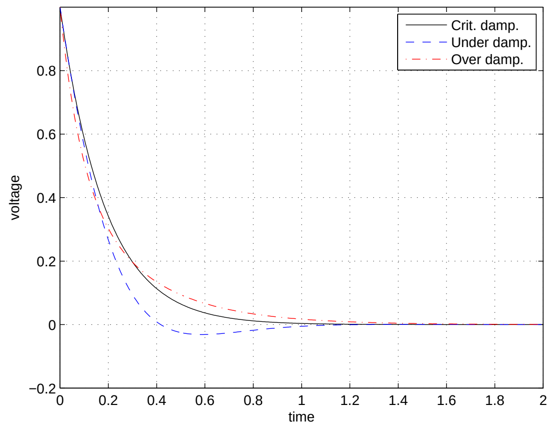

Figure 2: A voltage in critically-damped, under-damped and over-damped versions of a circuit.

5.2 Critically-damped circuits (2 marks)

In this experiment we will construct a circuit that, nominally, is critically damped. We will then perform experiments to determine whether or not the circuit that is constructed from your design is (approximately) critically damped.

1. Design a critically-damped series RLC circuit.
2. Select a circuit scenario in which you will be able to measure the natural response, and select a setting for the power supply that will mimic that scenario.
3. Sketch what you see on the oscilloscope.
4. Now repeat with the value of the resistor changed so that the damping ratio is about 20% smaller, and sketch what you see on the same graph.
5. Repeat with the value of the resistor changed so that the damping ratio is about 20% higher than in the critically-damped case. Again, sketch what you see on the same graph.
6. Your sketch should resemble Figure 2 to some degree. Clearly explain the features of your sketch and how they relate to the nature of the damping in the circuit. Is the circuit that you implemented based on your critically-damped design close to being critically damped in practice?

6 Lab Report (3 marks)

In your report, discuss any differences between the designed and estimated values of ω_0 and ζ in Section 5.1, and provide your sketches and a clear discussion relating to point 6 in Section 5.2.