FORCED RESPONSE OF A SECOND-ORDER CIRCUIT

- Recall the differential equation, with $a_2 = 1$.

$$\frac{d^2x(t)}{dt^2} + a_1 \frac{dx(t)}{dt} + a_0 x(t) = f(t) \qquad \blacksquare$$

- The solution is of the form

$$x(t) = x_n(t) + x_f(t)$$

- Since the left hand side of $\blacksquare$ $= 0$ for $x_n(t)$
  then the left hand side of $\blacksquare$ $= f(t)$ for $x_f(t)$

- What might we expect $x_f(t)$ to be?

  if $f(t)$ is a constant: Derivatives of a constant are zero, so perhaps $x_f(t)$ is a constant

  if $f(t)$ is an exponential: Derivatives of an exponential are also exponentials, perhaps $x_f(t)$ is exponential

  if $f(t)$ is a sinusoid: Derivatives of sinusoids are sinusoids, perhaps $x_f(t)$ is a sinusoid.

Hence, for some simple, but important forcing functions, $x_f(t)$ has the same shape as $f(t)$.

Forcing function, $f(t)$	$x_f(t)$
K	A
Kt	$At+B$
Kt^2	At^2+Bt+C
$K\sin\omega_0 t$	$A\sin\omega_0 t + B\cos\omega_0 t$
$Ke^{-\alpha t}$	$Ae^{-\alpha t}$

- We will use the initial/boundary conditions to determine A, B, C

- Sometimes you may find that $x_f(t)$ is proportional to a component of $x_n(t)$; (e.g. $f(t)$ = exponential case).
- In that case you must replace $x_f(t)$ above by $t^p x_f(t)$ where p is the smallest integer such that $x_f(t)$ is no longer proportional to any component of $x_n(t)$

SUMMARY:

COMPLETE RESPONSE OF A SECOND-ORDER CIRCUIT

Given $$\frac{d^2x(t)}{dt^2} + a_1 \frac{dx(t)}{dt} + a_0 x(t) = f(t)$$

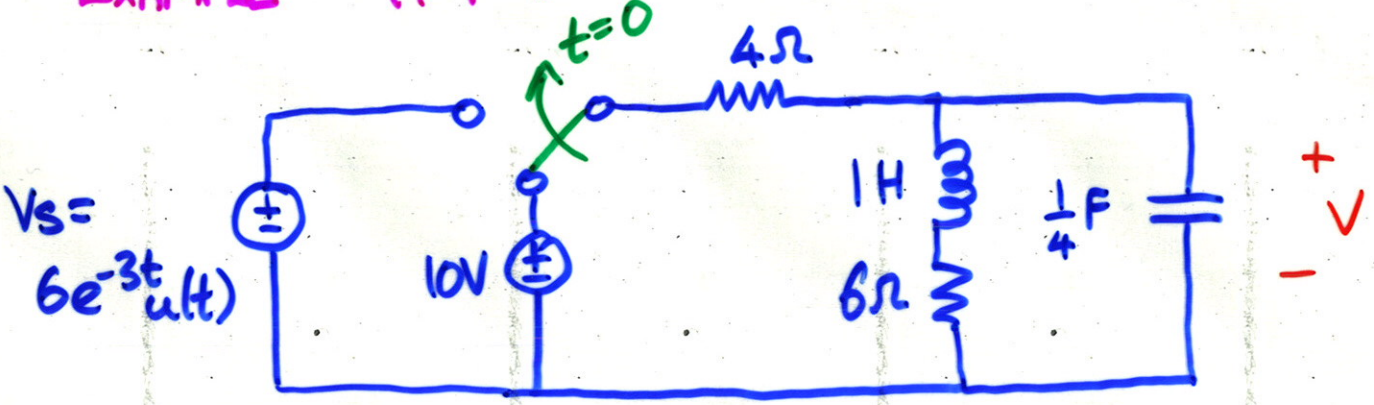
we have seen how to find the shape of

$$x(t) = x_n(t) + x_f(t)$$

1. Solve characteristic equation
$$s^2 + a_1 s + a_0 = 0$$
2. Use solutions to determine the shape of $x_n(t)$.
3. Determine the shape of $x_f(t)$ from the table. (i.e., $x_f(t)$ "looks" like $f(t)$)

- The resulting $x_n(t)$ and $x_f(t)$ contain unknown parameters.
- These can be determined by evaluating $x(0)$ and $\left.\frac{dx(t)}{dt}\right|_{t=0}$ and using knowledge from the circuit
- When the input $f(t)$ is a constant, $x(\infty)$ can also be determined + used here

EXAMPLE : 9.9-1



Find $v(t)$ for $t>0$, assuming circuit is in steady state at $t=0$

Note $u(t)$ isthe unit step function

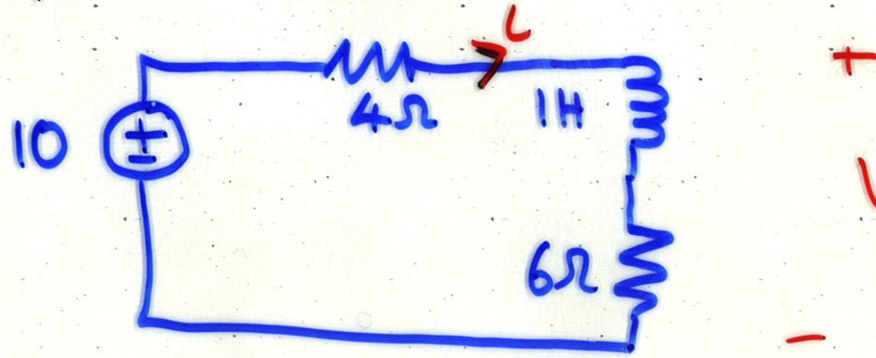
$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwize} \end{cases}$$

STEP 1 Determine initial conditions

at time $t=0^-$ circuit is in steadystate.

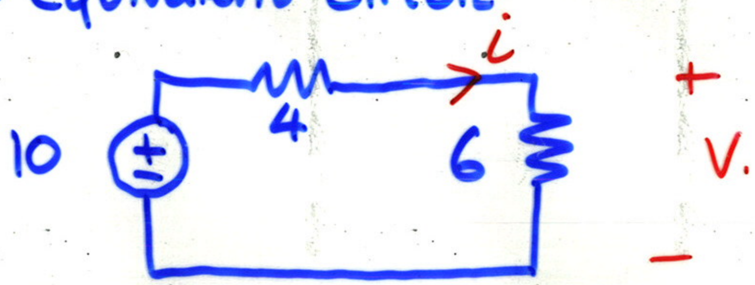
$\Rightarrow$ since source is constant, no current flows through the capacitor

Equivalent circuit



Since the source is constant, in steady state there is no voltage drop over inductor

New equivalent circuit



By voltage division, $V(0^-) = \frac{6}{10} \times 10$

$= 6V$

$i = i_L(0^-) \Rightarrow i_L(0^-) = 1A.$

$\Rightarrow$ Initial conditions are:

$V(0^-) = 6V$

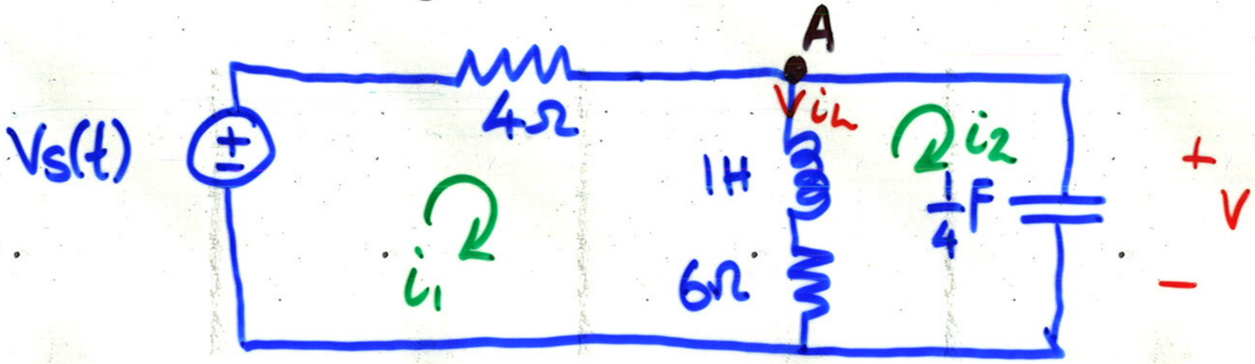
$i_L(0^-) = 1A$

Both these quantities are continuous

$\Rightarrow$ $V(0^+) = 6V$

$i_L(0^+) = 1A$

After switching we have:



Note: i_1, i_2 not used directly!

KVL loop 2

$$-V(t) + \frac{d i_L(t)}{dt} + 6\, i_L(t) = 0 \qquad ①$$

KCL, node A

$$\frac{V(t) - V_s(t)}{4} + i_L(t) + \frac{1}{4}\frac{dV(t)}{dt} = 0 \qquad ②$$

$$② \Rightarrow \quad i_L(t) = \frac{V_s(t) - V(t)}{4} - \frac{1}{4}\frac{dV(t)}{dt}$$

Substitute into ①

$$\Rightarrow \quad \frac{d^2 V(t)}{dt^2} + \frac{7\, dV(t)}{dt} + 10V = \frac{dV_s(t)}{dt} + 6V_s(t)$$

This is the differential equation we must solve

Find Natural Response

Characteristic equation is

$$s^2 + 7s + 10 = 0.$$

$\Rightarrow s = -2, \; s = -5 \Rightarrow$ overdamped circuit

$\Rightarrow V_n(t) = A_1 e^{-2t} + A_2 e^{-5t}$ for $t \geq 0$

Find Forced Response

Forcing function $f(t) = \dfrac{dV_s(t)}{dt} + 6V_s(t)$

$= 18e^{-3t}, \; t \geq 0$

$\Rightarrow V_f(t) = Be^{-3t}$

This is not proportional to any component of $V_n(t)$ so it does not need to be modified

Now we must find A_1, A_2, B.

Since the natural response vanishes (goes to zero) on the left hand side of the differential equation, we have that

$$\frac{d^2 v_f(t)}{dt^2} + 7\frac{dv_f(t)}{dt} + 10v_f(t) = 18e^{-3t}$$

$$\Rightarrow \quad 9Be^{-3t} - 21Be^{-3t} + 10Be^{-3t} = 18e^{-3t}.$$

$$\Rightarrow \quad B = -9.$$

Therefore

$$v(t) = A_1 e^{-2t} + A_2 e^{-5t} - 9e^{-3t}$$

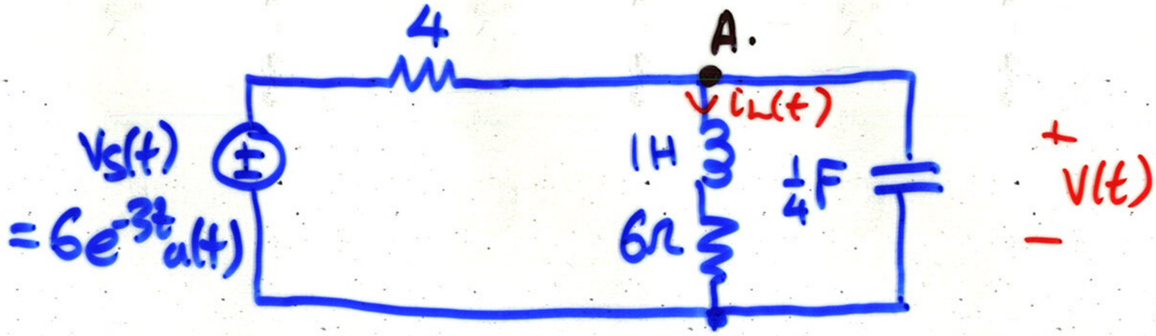
Now find A_1, A_2 from the facts that

$$v(0) = 6V$$
$$i_L(0) = 1A.$$

$$v(t) = A_1 e^{-2t} + A_2 \text{ [redacted] } e^{-5t} - 9e^{-3t}$$

$$\Rightarrow \quad v(0) = A_1 + A_2 - 9 = 6.$$

To find another equation we need to relate $v(t)$ to $i_L(t)$.



KCL at A, [redacted]
$$i_L(t) = \frac{v_s(t) - v(t)}{4} - \frac{1}{4}\frac{dv(t)}{dt}$$

$$\Rightarrow \quad i_L(0) = \frac{v_s(0)}{4} - \frac{v(0)}{4} - \frac{1}{4}\left.\frac{dv(t)}{dt}\right|_{t=0}$$

$$\Rightarrow \quad 1 = \frac{6}{4} - \frac{6}{4} - \frac{1}{4}(-2A_1 - 5A_2 + 27)$$

$$\Rightarrow \quad 2A_1 + 5A_2 = 31.$$

Thus we have:
$$\left.\begin{array}{l} A_1 + A_2 = 15 \\ 2A_1 + 5A_2 = 31. \end{array}\right\} \Rightarrow \begin{array}{l} A_1 = \frac{44}{3} \\ A_2 = 1/3 \end{array}$$

$$\Rightarrow v(t) = \frac{44}{3}e^{-2t} + \frac{1}{3}e^{-5t} - 9e^{-3t} \text{ for } t \geq 0$$

Sanity Checks.

1). $V_s(t)$ decays to zero, Does $v(t)$ decay to zero?

Yes!