EE3CL4: Introduction to Linear Control Systems

Section 8: Frequency Domain Techniques

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Outline

Transfer Functions: A Quick Review

- Consider a transfer function

$$G(s) = K \frac{\prod_i (s + z_i)}{\prod_j (s + p_j)}$$

- Zeros: $-z_i$; Poles: $-p_j$
- Note that $s + z_i = s - (-z_i)$,
- This is the vector from $-z_i$ to s
- Magnitude:

$$|G(s)| = |K| \frac{\prod_i |s + z_i|}{\prod_j |s + p_j|} = |K| \frac{\text{prod. dist's from zeros to } s}{\text{prod. dist's from poles to } s}$$

- Phase:

$$\angle G(s) = \angle K + \text{sum angles from zeros to } s \\ - \text{sum angles from poles to } s$$

Frequency Response

- For a stable, linear, time-invariant (LTI) system, the steady state response to a sinusoidal input is a sinusoid of the same frequency but possibly different magnitude and different phase
- Sinusoids are the eigenfunctions of convolution
- If input is $A\cos(\omega_0 t + \theta)$
  and steady-state output is $B\cos(\omega_0 t + \phi)$,
  then the complex number $B/Ae^{j(\phi-\theta)}$
  is called the frequency response of the system at frequency ω_0.

Frequency Response, II

- If a stable LTI system has a transfer function $G(s)$, then the frequency response at ω_0 is $G(s)|_{s=j\omega_0}$
- What if the system is unstable?

Plotting the frequency response

- For each ω, $G(j\omega)$ is a complex number.
- How should we plot it?

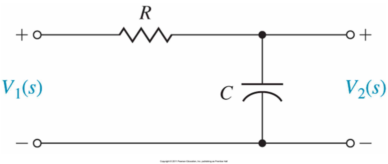
- $G(j\omega) = \left|G(j\omega)\right| e^{j\angle G(j\omega)}$
  Plot $\left|G(j\omega)\right|$ versus ω, and $\angle G(j\omega)$ versus ω

- Plot $20\log_{10}\left(\left|G(j\omega)\right|\right)$ versus $\log_{10}(\omega)$, and $\angle G(j\omega)$ versus $\log_{10}(\omega)$

- $G(j\omega) = \mathrm{Re}\big(G(j\omega)\big) + j\,\mathrm{Im}\big(G(j\omega)\big)$
  Plot the curve $\Big(\mathrm{Re}\big(G(j\omega)\big), \mathrm{Im}\big(G(j\omega)\big)\Big)$ on an "x–y" plot
  Equiv. to curve $\left|G(j\omega)\right| e^{j\angle G(j\omega)}$ as ω changes (polar plot)

Polar plot, example 1

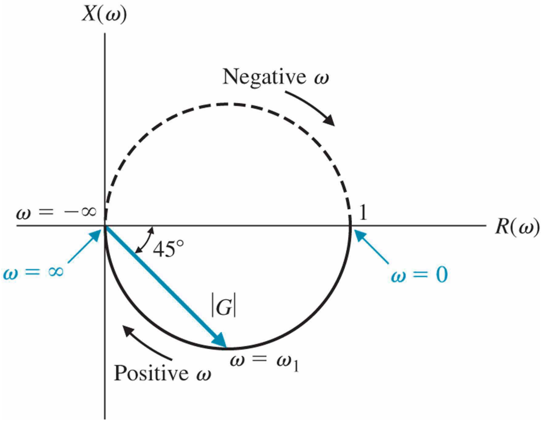
Let's consider the example of an RC circuit



- $G(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{1+sRC}$
- $G(j\omega) = \frac{1}{1+j\omega/\omega_1}$, where $\omega_1 = 1/(RC)$.
- $G(j\omega) = \frac{1}{1+(\omega/\omega_1)^2} - j\frac{\omega/\omega_1}{1+(\omega/\omega_1)^2}$
- $G(j\omega) = \frac{1}{\sqrt{1+(\omega/\omega_1)^2}}\, e^{-j\,\mathrm{atan}(\omega/\omega_1)}$

Polar plot, example 1

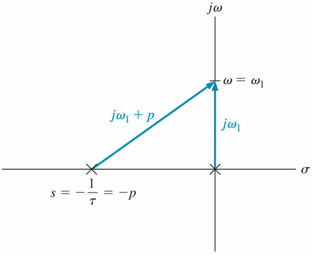
- $G(j\omega) = \frac{1}{1+(\omega/\omega_1)^2} - j\frac{\omega/\omega_1}{1+(\omega/\omega_1)^2}$
- $G(j\omega) = \frac{1}{\sqrt{1+(\omega/\omega_1)^2}}\, e^{-j\,\mathrm{atan}(\omega/\omega_1)}$



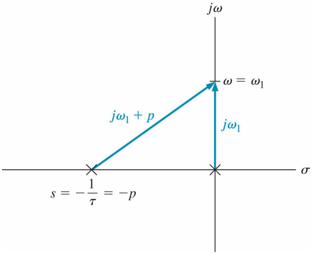
Polar plot, example 2

Consider $G(s) = \frac{K}{s(s\tau+1)}$.

- Poles at origin and $s = -1/\tau$.
- To use geometric insight to plot polar plot, rewrite as $G(s) = \frac{K/\tau}{s(s+1/\tau)}$
- Then $\left|G(j\omega)\right| = \frac{K/\tau}{|j\omega|\,|j\omega+1/\tau|}$ and $\angle G(j\omega) = -\angle(j\omega) - \angle(j\omega + 1/\tau)$

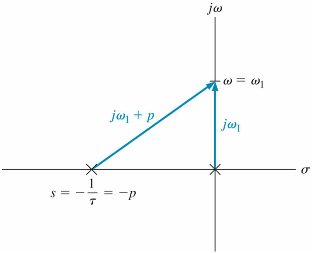


Polar plot, ex. 2, $G(s) = \frac{K/\tau}{s(s+1/\tau)}$



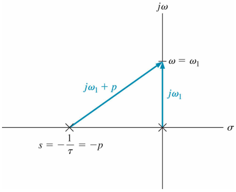
- When $\omega \to 0^+$, $|G(j\omega)| \to \infty$, $\angle G(j\omega) \to -90^\circ$ from below
  Tricky
- To get a better feel, write $G(j\omega) = \frac{-K\omega^2\tau}{\omega^2+\omega^4\tau^2} - j\frac{\omega K}{\omega^2+\omega^4\tau^2}$
  Hence, as $\omega \to 0^+$, $G(j\omega) \to -K\tau - j\infty$
- As ω increases, distances from poles to $j\omega$ increase.
  Hence $|G(j\omega)|$ decreases
- As ω increases, angle from pole at $-1/\tau$ increases.
  Hence $\angle G(j\omega)$ becomes more negative

Polar plot, ex. 2, $G(s) = \frac{K/\tau}{s(s+1/\tau)}$



- When $\omega = 1/\tau$, $G(j\omega) = (K/\tau)/\big((1/\tau)(\sqrt{2}/\tau)\big)\, e^{-j(90^\circ+45^\circ)}$ i.e., $G(j\omega)|_{\omega=1/\tau} = (K\tau/\sqrt{2})e^{-j135^\circ}$
- As ω approaches $+\infty$, both distances from poles get large. Hence $|G(j\omega)| \to 0$
- As ω approaches $+\infty$, angle from $-1/\tau$ approaches -90° from below. Hence $\angle G(j\omega)$ approaches -180° from below

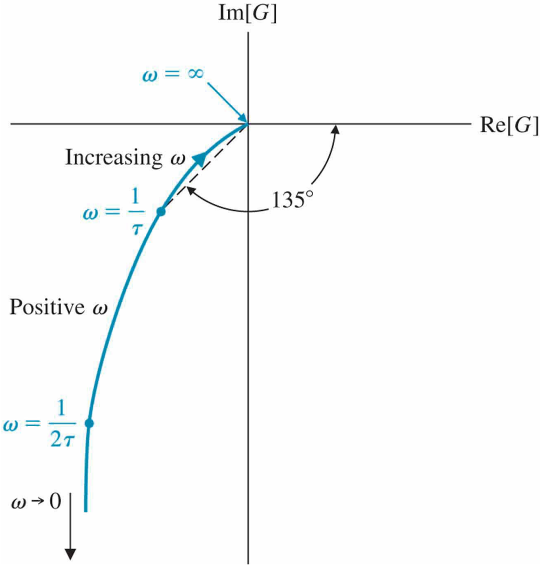
Polar plot, ex. 2, $G(s) = \frac{K/\tau}{s(s+1/\tau)}$



Summary

- As $\omega \to 0^+$, $G(j\omega) \to -K\tau - j\infty$
- As ω increases,
  $|G(j\omega)|$ decreases, $\angle G(j\omega)$ becomes more negative
- When $\omega = 1/\tau$, $G(j\omega) = (K/\sqrt{2})e^{-j135^\circ}$
- As ω approaches $+\infty$,
  $G(j\omega)$ approaches zero from angle -180°

Polar plot, ex. 2, $G(s) = \frac{K/\tau}{s(s+1/\tau)}$



Bode Diagrams

- Bode magnitude plot

$$20 \log_{10} |G(j\omega)| \quad \text{against} \quad \log_{10} \omega$$

- Bode phase plot

$$\angle G(j\omega) \quad \text{against} \quad \log_{10} \omega$$

- In 2CJ4 we developed rules to help sketch these plots
- In this course we will use these sketches to design controllers

Sketching Bode Diagrams

- Consider generic transfer function of LTI system

$$G(s) = \frac{K \prod_i (s + z_i) \prod_k (s^2 + 2\zeta_k \omega_{n,k} s + \omega_{n,k}^2)}{s^N \prod_j (s + p_j) \prod_r (s^2 + 2\zeta_{d,k} \omega_{nd,r} s + \omega_{nd,r}^2)}$$

  where z_i and p_j are real.
- Unfortunately, not in the form that we are used to for Bode diagrams
- Divide numerator by $\prod_i z_i \prod_k \omega_{n,k}^2$
- Similarly for denominator
- Then if $\tilde{K} = K \prod_i z_i \prod_k \omega_{n,k}^2 / \big(\prod_j p_j \prod_r \omega_{nd,r}^2\big)$,

$$G(s) = \frac{\tilde{K} \prod_i (1 + s/z_i) \prod_k \big(1 + 2\zeta_k (s/\omega_{n,k}) + (s/\omega_{n,k})^2\big)}{s^N \prod_j (1 + s/p_j) \prod_r \big(1 + 2\zeta_{d,k} (s/\omega_{nd,r}) + (s/\omega_{nd,r})^2\big)}$$

Sketching Bode Diagrams, II

- Now, frequency response can be written as:

$$G(j\omega) = \frac{\tilde{K}\prod_i(1 + j\omega/z_i)}{(j\omega)^N\prod_j(1 + j\omega/p_j)} \times \frac{\prod_k\big(1 + 2\zeta_k(j\omega/\omega_{n,k}) + (j\omega/\omega_{n,k})^2\big)}{\prod_r\big(1 + 2\zeta_{d,k}(j\omega/\omega_{nd,r}) + (j\omega/\omega_{nd,r})^2\big)}$$

- Four key components:
  - Gain, $\tilde{K}$
  - Poles (or zeros) at origin
  - Poles and zeros on real axis
  - Poles and zeros in complex conjugate pairs
- Each contributes to the Bode Diagram

Bode Magnitude diagram

$$G(j\omega) = \frac{\tilde{K}\prod_i(1+j\omega/z_i)}{(j\omega)^N\prod_j(1+j\omega/p_j)} \times \frac{\prod_k\big(1+2\zeta_k(j\omega/\omega_{n,k})+(j\omega/\omega_{n,k})^2\big)}{\prod_r\big(1+2\zeta_{d,k}(j\omega/\omega_{nd,r})+(j\omega/\omega_{nd,r})^2\big)}$$

- Bode Magnitude diagram:

$$20\log_{10}|G(j\omega)| \text{ against } \log_{10}\omega$$

- $20\log_{10}|G(j\omega)|$ is

Sum of $20\log_{10}$ of components of numerator
– sum of $20\log_{10}$ of components of denominator

Components for magnitude

$$G(j\omega) = \frac{\tilde{K}\prod_i(1 + j\omega/z_i)}{(j\omega)^N \prod_j(1 + j\omega/p_j)} \times \frac{\prod_k\left(1 + 2\zeta_k(j\omega/\omega_{n,k}) + (j\omega/\omega_{n,k})^2\right)}{\prod_r\left(1 + 2\zeta_{d,k}(j\omega/\omega_{nd,r}) + (j\omega/\omega_{nd,r})^2\right)}$$

- Poles at origin: slope starts at $-20N$ dB/dec
- Gain $|\tilde{K}|$ incorporated in position of that sloping line
- First order component in numerator:
  increase slope by 20 dB/dec at $\omega = z_i$
- First order component in denominator:
  decrease slope by 20 dB/dec at $\omega = p_j$
- Second order components:
  increase or decrease slope by 40 dB/dec at $\omega = \omega_n$

Bode Phase Diagram

$$G(j\omega) = \frac{\tilde{K}\prod_i(1 + j\omega/z_i)}{(j\omega)^N \prod_j(1 + j\omega/p_j)} \times \frac{\prod_k\big(1 + 2\zeta_k(j\omega/\omega_{n,k}) + (j\omega/\omega_{n,k})^2\big)}{\prod_r\big(1 + 2\zeta_{d,k}(j\omega/\omega_{nd,r}) + (j\omega/\omega_{nd,r})^2\big)}$$

- Bode Phase Diagram

  $\angle G(j\omega)$ against $\log_{10}\omega$

- $\angle G(j\omega)$ is

  Sum of phases of components of numerator

  – sum of phases of components of denominator

Components

$$G(j\omega) = \frac{\tilde{K}\prod_i (1 + j\omega/z_i)}{(j\omega)^N \prod_j (1 + j\omega/p_j)} \times \frac{\prod_k \left(1 + 2\zeta_k (j\omega/\omega_{n,k}) + (j\omega/\omega_{n,k})^2\right)}{\prod_r \left(1 + 2\zeta_{d,k} (j\omega/\omega_{nd,r}) + (j\omega/\omega_{nd,r})^2\right)}$$

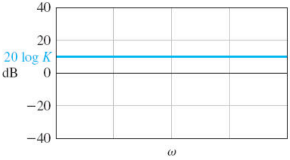
- Phase of $\tilde{K}$
- Poles at origin: $-N90^\circ$
- First order component in numerator:
  linear phase change of $+90^\circ$ over $\omega \in [z_i/10, 10z_i]$
- First order component in denominator:
  linear phase change of -90° over $\omega \in [p_j/10, 10p_j]$
- Second order components:
  phase change of $\pm 180^\circ$ around $\omega = \omega_n$

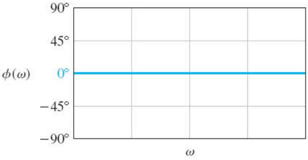
Graphically

Table 8.3 Asymptotic Curves for Basic Terms of a Transfer Function

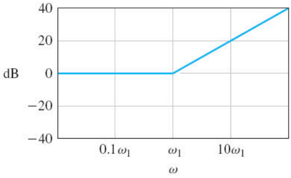
Term — Magnitude $20 \log|G|$ — Phase $\phi(\omega)$

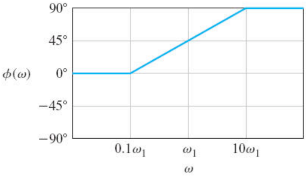
1. Gain, $G(j\omega) = K$



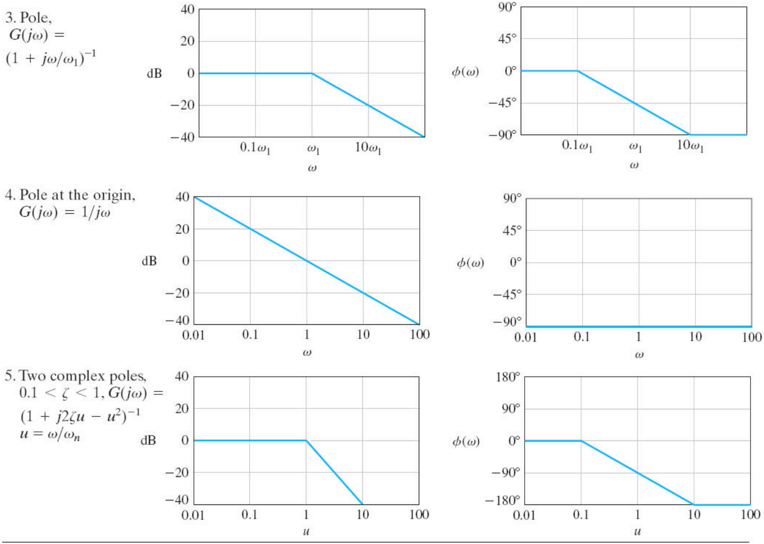


2. Zero, $G(j\omega) = 1 + j\omega/\omega_1$



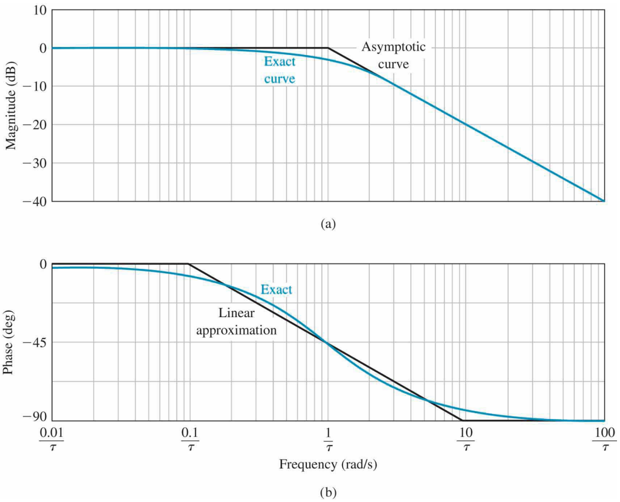


Graphically



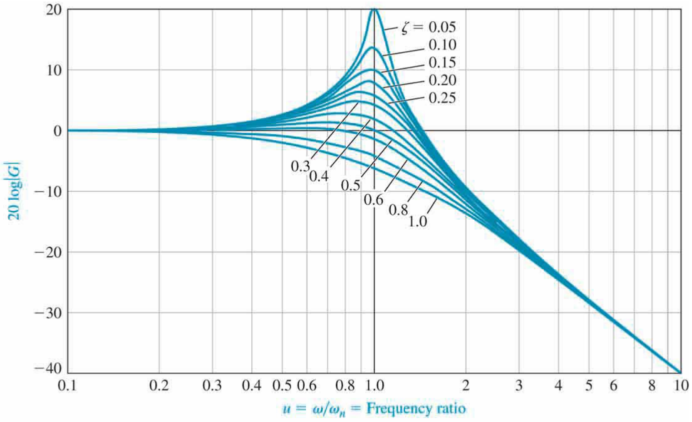
Accuracy of Bode Sketches

Isolated first order pole (analogous for zero)



Accuracy of Bode Sketches

Isolated complex conjugate pair of poles



Accuracy of Bode Sketches

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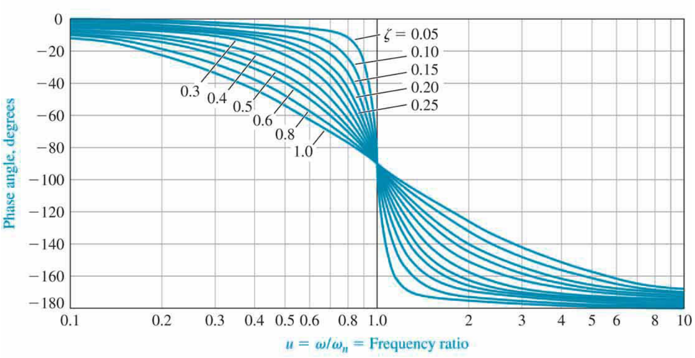
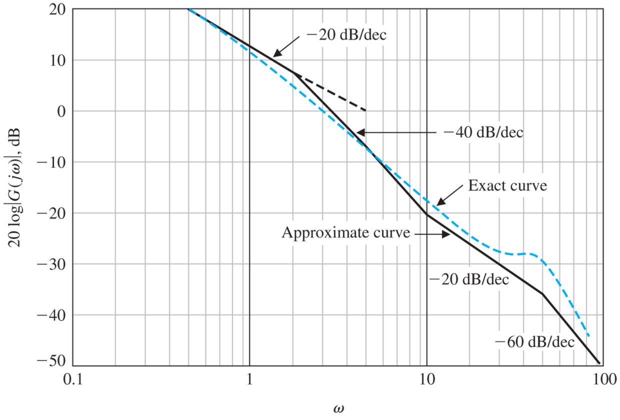


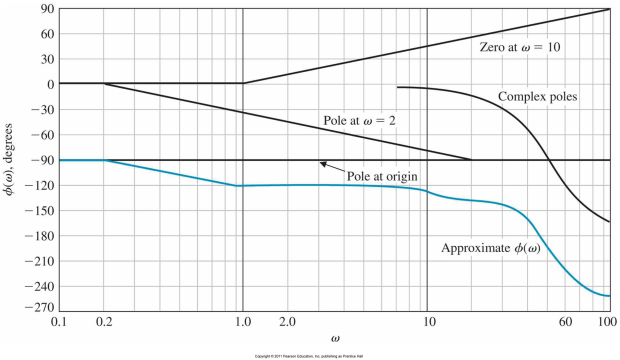
Example

$$G(j\omega) = \frac{5(1 + j\omega/10)}{j\omega(1 + j\omega/2)\big(1 + 0.6(j\omega/50) + (j\omega/50)^2\big)}$$



Example

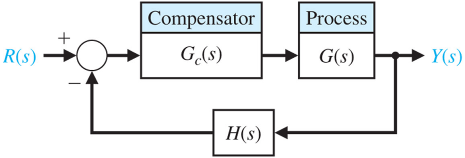
$$G(j\omega) = \frac{5(1 + j\omega/10)}{j\omega(1 + j\omega/2)\big(1 + 0.6(j\omega/50) + (j\omega/50)^2\big)}$$



Introduction

- We have seen techniques that determine stability of a system:
  - Routh-Hurwitz
  - root locus
- However, both of them require a model for the plant
- Today: frequency response techniques
  - Although they work best with a model
  - For an open-loop stable plant, they also work with measurements
- Key result: Nyquist's stability criterion
- Design implications: Bode techniques based on gain margin and phase margin

Characteristic equation



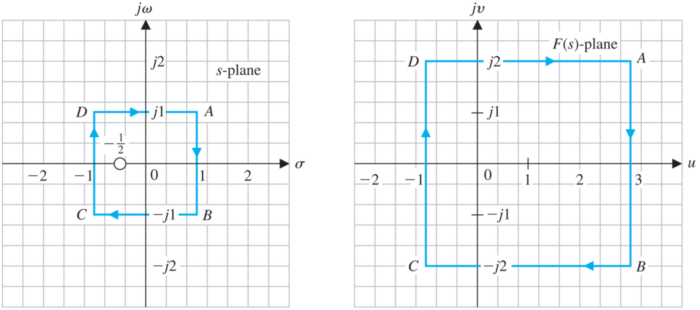
- To determine the stability of the system we need to examine the characteristic equation:

$$F(s) = 1 + L(s) = 0$$

where $L(s) = G_c(s)G(s)H(s)$.

- The key result involves mapping a closed contour of values of s to a closed contour of values of $F(s)$.
- We will investigate the idea of mappings first

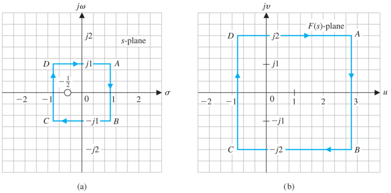
Simple example



(a) (b)

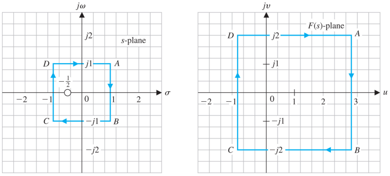
- Set $F(s) = 2s + 1$
- Map the square in the "s-plane" to the contour in the "$F(s)$-plane"

Area enclosed



- How might we define area enclosed by a closed contour?
- We will be perfectly rigorous, but will go against mathematical convention
- Define area enclosed to be that to the right when the contour is traversed clockwise
- What you see when moving clockwise with eyes right
- Sometimes we say that this area is the area "inside" the clockwise contour
- Notions of "enclosed" or "inside" will be applied to contours in the *s*-plane

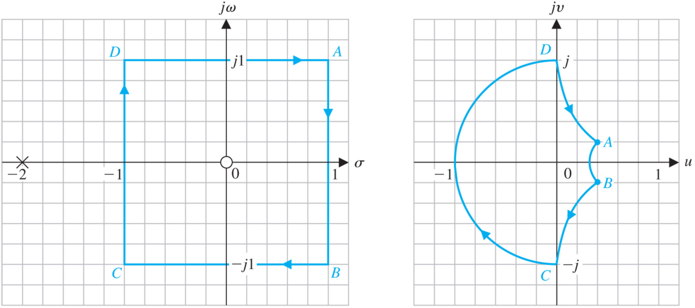
Encirclement



(a) (b)

- In the $F(s)$-plane, we will be interested in the notion of encirclement of the origin
- A contour is said to encircle the origin in the clockwise direction, if the contour completes a 360° revolution around the origin in the clockwise direction.
- A contour is said to encircle the origin in the anti-clockwise direction, if the contour completes a 360° revolution around the origin in the anti-clockwise direction.
- We will say that an anti-clockwise encirclement is a "negative" clockwise encirclement

Example with rational $F(s)$



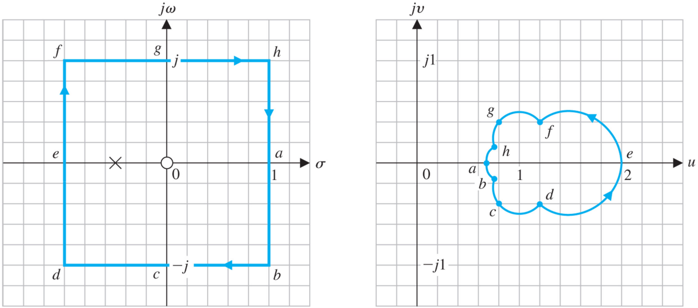
(a) (b)

- A mapping for $F(s) = \frac{s}{s+2}$
- Note that s-plane contour encloses the zero of $F(s)$
- How many times does the $F(s)$-plane contour encircle the origin in the clockwise direction?

Cauchy's Theorem

- Nyquist's Criterion is based on Cauchy's Theorem:
  - Consider a rational function $F(s)$
  - If the clockwise traversal of a contour Γ_s in the s-plane encloses Z zeros and P poles of $F(s)$
    and does not go through any poles or zeros
  - then the corresponding contour in the $F(s)$-plane, Γ_F encircles the origin $N = Z - P$ times in the clockwise direction
- A sketch of the proof later.
- First, some examples

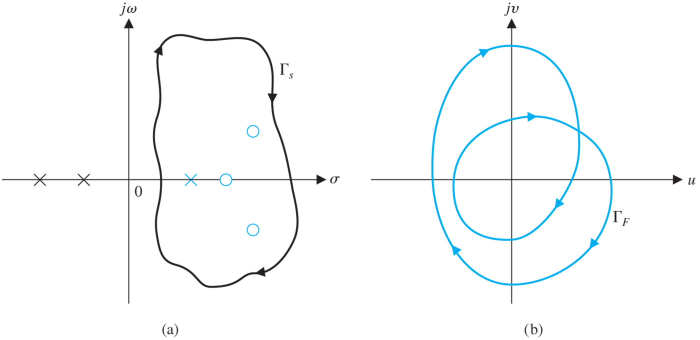
Example 1



(a) (b)

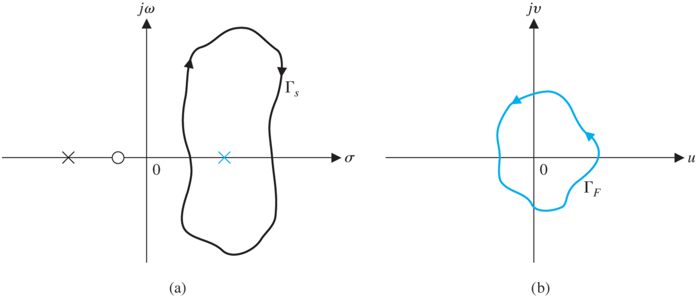
- A mapping for $F(s) = \frac{s}{s+1/2}$
- s-plane contour encloses a zero and a pole
- Theorem suggests no clockwise encirclements of origin of $F(s)$-plane
- This is what we have!

Example 2



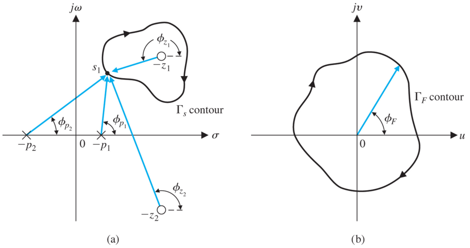
- s-plane contour encloses 3 zeros and a pole
- Theorem suggests 2 clockwise encirclements of the origin of the $F(s)$-plane

Example 3



- *s*-plane contour encloses one pole
- Theorem suggests -1 clockwise encirclements of the origin of the $F(s)$-plane
- That is, one anti-clockwise encirclement

Informal Justification of Cauchy's Theorem



- Consider the case of $F(s) = \frac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)}$
- $\angle F(s_1) = \phi_{z_1} + \phi_{z_2} - \phi_{p_1} - \phi_{p_2}$
- As the contour is traversed the nett contribution from ϕ_{z_1} is 360 degrees
- As contour is traversed, the nett contribution from other angles is 0 degrees
- Hence, as contour is traversed, $\angle F(s)$ changes by 360 degrees. One encirclement!

Informal Justification

- Extending this to any number of poles and zeros inside the contour
- For a closed contour, the change in $\angle F(s)$ is $360Z - 360P$
- Hence $F(s)$ encircles origin $Z - P$ times

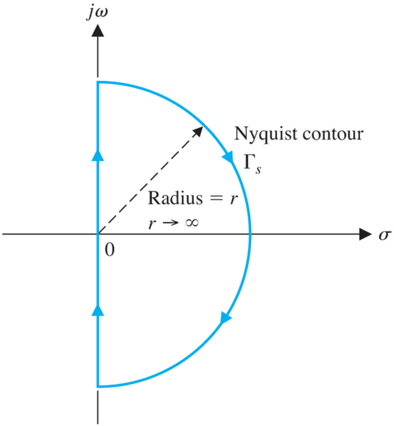
Cauchy's Theorem (Review)

- Consider a rational function $F(s)$
- If the clockwise traversal of a contour Γ_s in the s-plane encloses Z zeros and P poles of $F(s)$ and does not go through any poles or zeros
- then the corresponding contour in the $F(s)$-plane, Γ_F encircles the origin $N = Z - P$ times in the clockwise direction

Nyquist's goal

- Nyquist was concerned about testing for stability
- How might one use Cauchy Theorem to examine this?
- Perhaps choose $F(s) = 1 + L(s)$, as this determines stability
- Which contour should we use?

Nyquist's contour

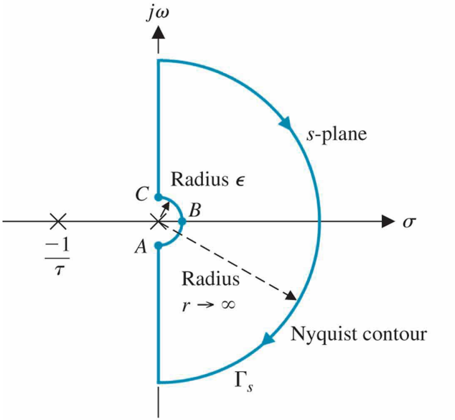


Actually, we have to be careful regarding poles and zeros on the $j\omega$-axis, including the origin.

Standard approach is to indent contour so that it goes to the right of any such poles or zeros

Modified Nyquist contour

Here's an example for a model like that of the motor in the lab.



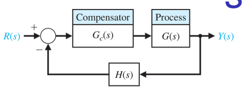
Coarse Applic. of Cauchy

- Recall that the zeros of $F(s) = 1 + L(s)$ are the poles of the closed loop
- Let P denote the number of right half plane poles of $F(s)$
- The number of right half plane zeros of $F(s)$ is $N + P$, where N is the number of clockwise encirclements of the origin made by the image of Nyquist's contour in the $F(s)$ plane.
- A little difficult to parse.
- Perhaps we can apply Cauchy's Theorem in a more sophisticated way.

Towards Nyquist's Criterion

- $F(s) = 1 + L(s)$, where $L(s)$ is the open loop transfer function
- Encirclement of the origin in $F(s)$-plane is the same as encirclement of -1 in the $L(s)$-plane
- This is more convenient, because $L(s)$ is often factorized, and hence we can easily determine P
- Now that we are dealing with $L(s)$, P is the number of right-half plane poles of the open loop transfer function
- If we handle the remainder of the components of Cauchy's theorem carefully we obtain:

Nyquist's Criterion: Simplified statement

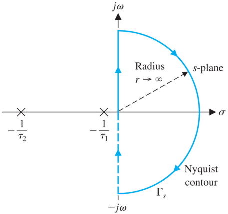


- Consider a unity feedback system with an open loop transfer function $L(s) = G_c(s)G(s)H(s)$,with no z's or p's on $j\omega$-axis
- Let P_L denote the number of poles of $L(s)$ in RHP
- Consider the Nyquist Contour in the s-plane
- Let Γ_L denote image of Nyquist Contour under $L(s)$
- Let N_L denote the number of clockwise encirclements that Γ_L makes of the point $(-1, 0)$
- **Nyquist's Stability Criterion:**

$$\text{Number of closed-loop poles in RHP} = N_L + P_L$$

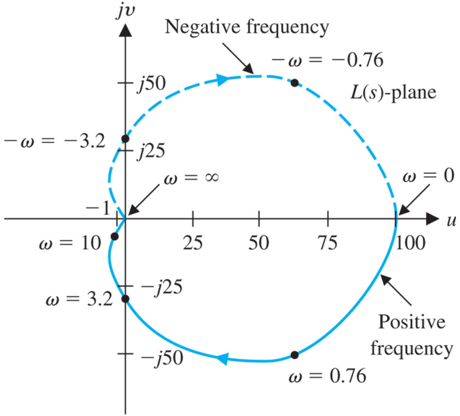
- Note that for a stable open loop, the closed-loop is stable if the image of the Nyquist contour does not encircle $(-1, 0)$.

Ex: $L(s) = \frac{1000}{(s+1)(s+10)}$ (stable)



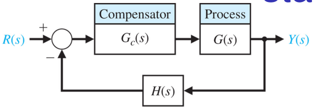
- For $0 \leq \omega < \infty$:
  - No zeros, two poles.
  - $|L(0)| = 1000/(1 \times 10) = 100$; $\angle L(0) = -0 - 0 = 0$
  - Distances from poles to $j\omega$ is increasing; hence $|L(j\omega)|$ is decreasing
  - Angles from poles to $j\omega$ are increasing; hence $\angle L(j\omega)$ is decreasing
  - As $\omega \to \infty$, $|L(j\omega)| \to 0$, $\angle L(j\omega) \to -180°$
- Recall that $L(-j\omega) = L(j\omega)^*$
- Remember to examine the $r \to \infty$ part of the curve

Ex: $L(s) = \frac{1000}{(s+1)(s+10)}$ (stable)



Note: No encirclements of $(-1, 0) \implies$ closed loop is stable

Nyquist's Criterion: Refined statement



- Consider a unity feedback system with an open loop transfer function $L(s) = G_c(s)G(s)H(s)$,
- Let P_L denote the number of poles of $L(s)$ in open RHP
- Consider the modified Nyquist Contour in the s-plane looping to the right of any poles or zeros on the $j\omega$-axis
- Let Γ_L denote image of mod. Nyquist Contour under $L(s)$
- Let N_L denote the number of clockwise encirclements that Γ_L makes of the point $(-1, 0)$
- **Nyquist's Stability Criterion:**

  Number of closed-loop poles in open RHP $= N_L + P_L$

- Now we can handle open-loop poles and zeros on $j\omega$-axis

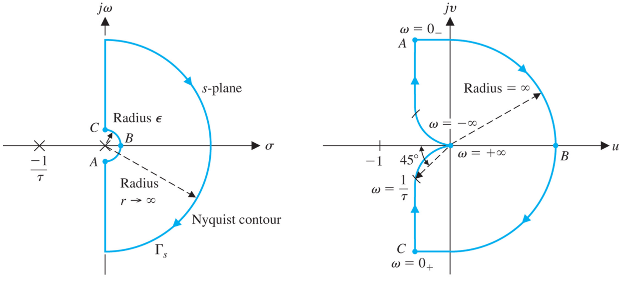
Example: Pole of $L(s)$ at origin

- Consider

$$L(s) = \frac{K}{s(\tau s + 1)}$$

- Like in servomotor
- Problem with the original Nyquist contour
- It goes through a pole!
- Cauchy's Theorem does not apply
- Must modify Nyquist Contour to go around pole
- Then Nyquist Criterion can be applied

Example: Pole of $L(s)$ at origin

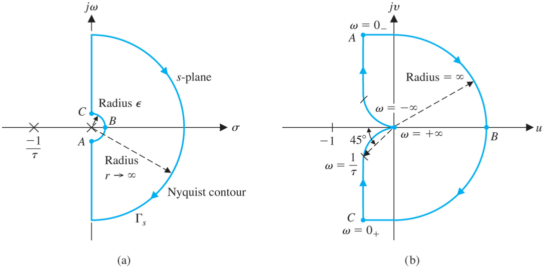


(a) (b)

Now three key aspects of the curve

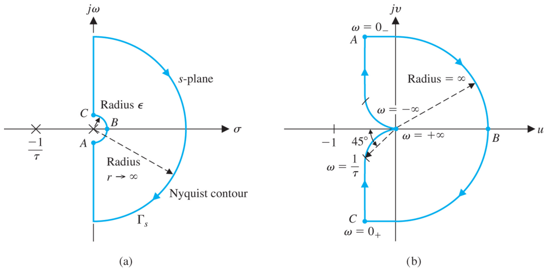
- Around the origin
- Positive frequency axis;
  remember negative freq. axis yields conjugate
- At ∞

Around the origin



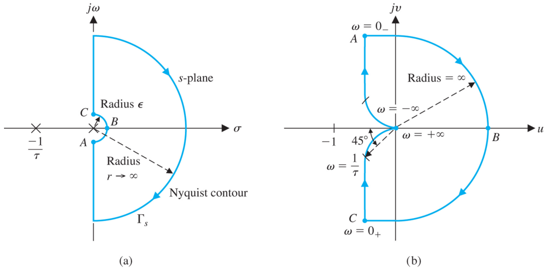
- $L(s) = \frac{K}{s(\tau s+1)}$
- Around the origin, $s = \epsilon e^{j\phi}$, where ϕ goes from -90° to 90°
- In the $L(s)$ plane: $\lim_{\epsilon\to 0} L(\epsilon e^{j\phi})$
- This is: $\lim_{\epsilon\to 0} \frac{K}{\epsilon e^{j\phi}} = \lim_{\epsilon\to 0} \frac{K}{\epsilon} e^{-j\phi}$

Up positive $j\omega$-axis



- For $0 < \omega < \infty$, $L(j\omega) = \frac{K}{\omega\sqrt{1+\omega^2\tau^2}} e^{-j(90^\circ + \text{atan}(\omega\tau))}$
- For small ω, $L(j\omega)$ is large with phase -90° Actually, as we worked out in a previous lecture, as $\omega \to 0^+$, $L(j\omega) \to -K\tau - j\infty$
- For large ω, $L(j\omega)$ is small with phase -180°
- For $\omega = 1/\tau$, $L(j\omega) = K\tau/\sqrt{2}\, e^{-j135^\circ}$

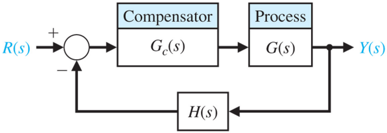
For $s = re^{j\theta}$ for large r



- For $s = re^{j\theta}$ with large r, and θ from $+90°$ to $-90°$,
- $\lim_{r\to\infty} L(re^{j\theta}) = \frac{K}{\tau r^2} e^{-j2\theta}$

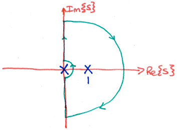
- How many encirclements of -1 in $L(s)$ plane? None
- Implies that closed loop is stable for all positive K
- Consistent with what we know from root locus (Lab. 2)

Example with open loop RHP pole, proportional control



- Consider $G(s) = \frac{1}{s(s-1)}$
- Essentially the same as plant model for VTOL aircraft example in root locus section
- Consider prop. control, $G_c(s) = K_1$, and $H(s) = 1$.
- Hence, $L(s) = \frac{K_1}{s(s-1)}$
- Observe that $L(s)$ has a pole in RHP; hence $P_L = 1$

Ex. with open loop RHP pole

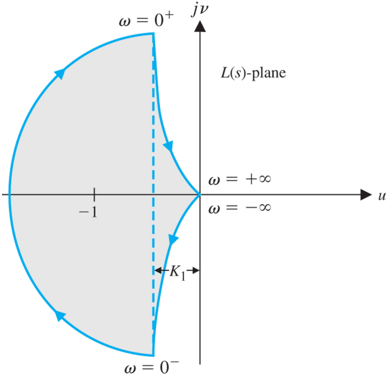


- $L(s) = \frac{K_1}{s(s-1)}$. For $s = j\omega$ and $0 < \omega < \infty$,

$$L(j\omega) = \frac{-K_1}{1+\omega^2} + j\frac{K_1}{\omega(1+\omega^2)} = \frac{K_1}{\omega\sqrt{1+\omega^2}}\angle\big(90^\circ + \mathrm{atan}(\omega)\big)$$

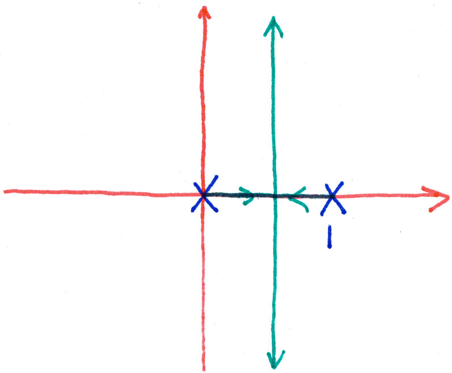
- For $\omega \to 0^+$, $L(j\omega) \to -K_1 + j\infty$
- As ω increases, real and imag. parts decrease, imag. part decreases faster
- Equiv. magnitude decreases, phase increases
- For $\omega \to \infty$, $L(j\omega)$ is small with angle $+180^\circ$
- Conjugate for $-\infty < \omega < 0$
- What about when $s = \epsilon e^{j\theta}$ for $-90^\circ \leq \theta \leq 90^\circ$? $L(s) = \frac{K_1}{\epsilon}\angle(-180^\circ - \theta)$

Example with open loop RHP pole

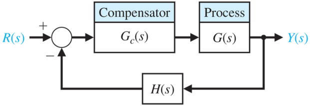


- Recall $P_L = 1$
- Number clockwise encirclements of $(-1, 0)$ is 1
- Hence there are two closed loop poles in the RHP for all positive values of K_1
- Consistent with root locus analysis

Root locus of $L(s) = \frac{1}{s(s-1)}$



Example with open loop RHP pole, PD control

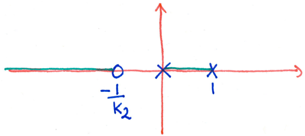


- $G(s) = \frac{1}{s(s-1)}$ and $H(s) = 1$. $L(s) = G_C(s)G(s)$.
- In the VTOL aircraft example, showed that closed-loop can be stabilized by lead compensation, $G_C(s) = \frac{K_c(s+z)}{(s+p)}$
- It can also be stabilized by PD comp., $G_C(s) = K_1(1 + K_2 s)$. (Under the presumption that this can be realized. It can be realized when we have "velocity" feedback.)
- Using the root locus, we can show that when $K_2 > 0$ there is a $K_1 > 0$ that stabilizes the closed loop (see next page)
- Can we see that in the Nyquist diagram?
- Plot the Nyquist diagram of $L(s) = G_c(s)G(s)$, where $G(s) = \frac{K_1}{s(s-1)}$ and $G_c(s) = 1 + K_2 s$

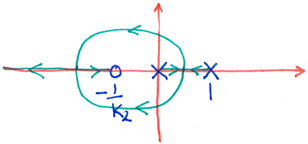
Root locus analysis

Root locus of $(1 + K_2 s)\frac{1}{s(s-1)}$ for a given $K_2 > 0$

- Poles, zero and active sections of real axis

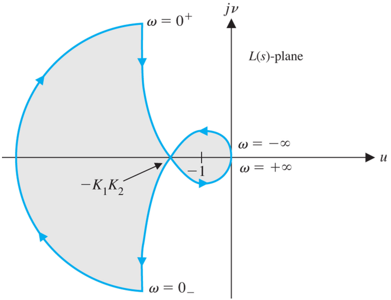


- Complete root locus



Conclusion: For any given $K_2 > 0$ there is a $\bar{K}_1 > 0$ such that closed loop is stable for all $K_1 > \bar{K}_1$. We can find $\bar{K}_1$ using Routh-Hurwitz

Nyquist diagram of $(1 + K_2 s)\frac{K_1}{s(s-1)}$



- Recall that $P_L = 1$
- If $K_1 K_2 > 1$, there is one anti-clockwise encirc. of -1
- In that case, number closed-loop poles in RHP is $-1 + 1 = 0$ and the closed loop is stable
- Consistent with root locus analysis; but gives $\bar{K}_1 = 1/K_2$ directly

One more example

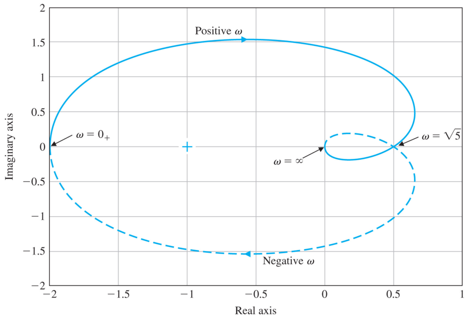
$$L(s) = \frac{K(s-2)}{(s+1)^2}$$

Open loop is stable, but has non-minimum phase (RHP) zero

$$L(j\omega) = \frac{K\sqrt{\omega^2+4}}{\omega^2+1}\angle\big(180^\circ - \text{atan}(\omega/2) - 2\,\text{atan}(\omega)\big)$$

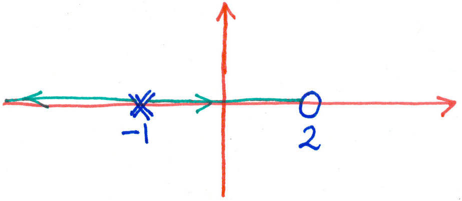
- For small positive ω, $L(j\omega) \approx 2K\angle 180^\circ$
- For large positive ω, $L(j\omega) \approx \frac{K}{\omega}\angle -90^\circ$
- In between, phase decreases monotonically, $180^\circ \to -90^\circ$. magnitude decreases monotonically (Bode mag dia.)
- $L(j\omega) = \frac{2K\left(2\omega^2-1+j\omega(5-\omega^2)\right)}{(1+\omega^2)^2}$; When $\omega = \sqrt{5}$, $L(j\omega) = K/2$
- When $s = re^{j\theta}$ with $r \to \infty$ and $\theta : 90^\circ \to -90^\circ$, $L(s) \to (K/r)e^{-j\theta}$

Nyquist plot of $L(s)/K$

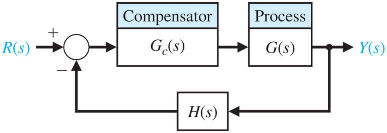


- Number of open loop RHP poles: 0
- Number of clockwise encirclements of -1:
  if $K < 1/2$: 0; if $K > 1/2$: 1
- Hence closed loop is
  stable for $K < 1/2$; unstable for $K > 1/2$
- This is what we would expect from root locus

Root locus of $L(s) = \frac{s-2}{(s+1)^2}$



Nyquist's Criterion (Review)



- Consider a unity feedback system with an open loop transfer function $L(s) = G_c(s)G(s)H(s)$,
- Let P_L denote the number of poles of $L(s)$ in open RHP
- Consider the modified Nyquist Contour in the s-plane (looping to the right of any poles or zeros on the $j\omega$-axis)
- Let Γ_L denote image of mod. Nyquist Contour under $L(s)$
- Let N_L denote the number of clockwise encirclements that Γ_L makes of the point $(-1, 0)$
- **Nyquist's Stability Criterion:**

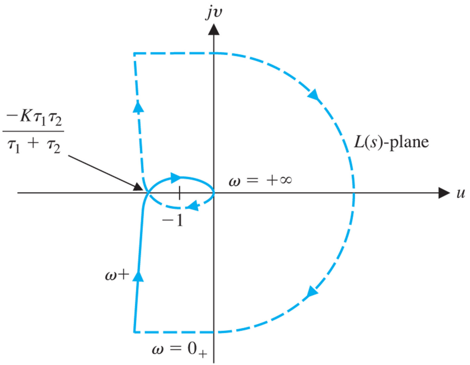
$$\text{Number of closed-loop poles in open RHP} = N_L + P_L$$

Relative Stability: Introductory Example

Consider

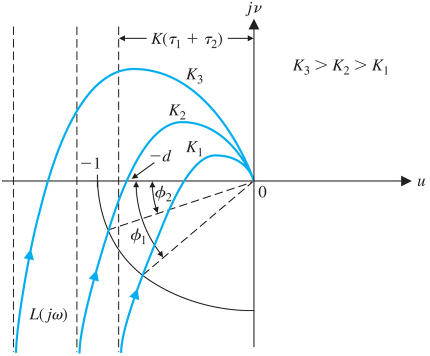
$$L(s) = \frac{K}{s(\tau_1 s + 1)(\tau_2 s + 1)}$$

Nyquist Diagram:



Zoom in

Since $L(s)$ is minimum phase (no RHP zeros), we can zoom in



For a given K,

- how much extra gain would result in instability?
  we will call this the gain margin
- how much extra phase lag would result in instability?
  we will call this the phase margin

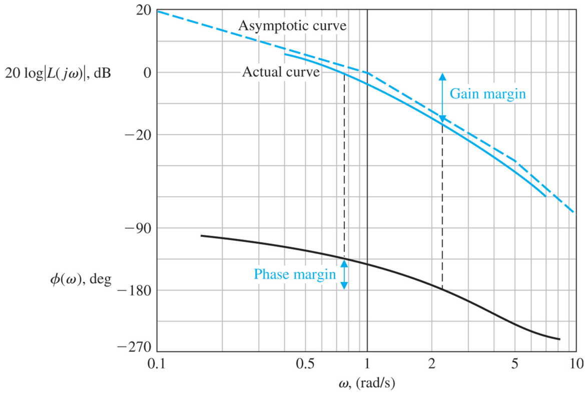
Formal definitions

- **Gain margin:** $\frac{1}{|L(j\omega_x)|}$,
  where ω_x is the frequency at which $\angle L(j\omega)$ reaches -180°
  amplifying the open-loop transfer function by this amount would result in a marginally stable closed loop

- **Phase margin:** $180^\circ + \angle L(j\omega_c)$,
  where ω_c is the frequency at which $|L(j\omega)|$ equals 1
  adding this much phase lag would result in a marginally stable closed loop

- These margins can be read from the Bode diagram

Bode diagram



$$L(j\omega) = \frac{1}{j\omega(1 + j\omega)(1 + j\omega/5)}$$

- Gain margin $\approx$ 15 dB
- Phase margin $\approx 43^\circ$

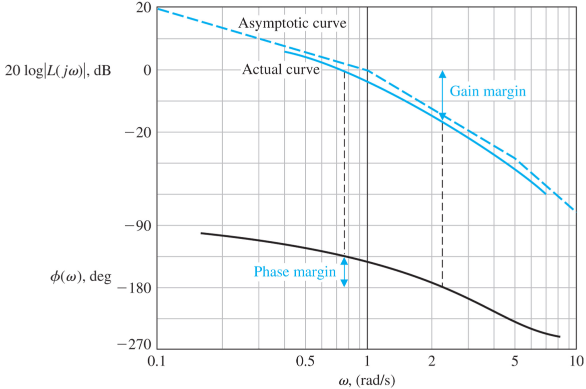
Phase margin and damping

- Consider a second-order open loop of the form $L(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$, with $\zeta < 1$
- Closed-loop poles $s_1, s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$
- Let ω_c be the frequency at which $|L(j\omega)| = 1$
- Square and rearrange: $\omega_c^4 + 4\zeta^2\omega_n^2\omega_c^2 - \omega_n^4 = 0$; Equivalently, $\frac{\omega_c^2}{\omega_n^2} = \sqrt{4\zeta^4+1} - 2\zeta^2$
- By definition, $\phi_{pm} = 180^\circ + \angle L(j\omega_c)$
- Hence

$$\phi_{pm} = \operatorname{atan}\left(\frac{2}{\sqrt{(4+1/\zeta^4)^{1/2}-2}}\right)$$

- Phase margin is an explicit function of damping ratio!
- Approximation: for $\zeta < 0.7$, $\zeta \approx 0.01\phi_{pm}$, where ϕ_{pm} is measured in degrees

Previous example



$$L(j\omega) = \frac{1}{j\omega(1 + j\omega)(1 + j\omega/5)}$$

- Phase margin $\approx 43^\circ$
- Damping ratio ≈ 0.43