TIME-DOMAIN INPUT-OUTPUT DESCRIPTIONS OF LTI SYSTEMS: DIFFERENTIAL AND DIFFERENCE EQUATIONS.

- In 2CJ4 we described the input and output of linear circuits in terms of differential equations
- So far we have only looked at convolution in 3TP4
- How are they related?
  (i) Impulse response assumes zero initial conditions inside the box (input is zero for a long time before the spike. In contrast differential equation approaches can handle non-zero initial conditions (these are rather uncommon in most systems applications)

  (ii) Convolution tends to be easier than solving the differential equation, Esp for orders higher than two

- Can we obtain the impulse response from the solution of differential equations?

Differential equations

Description is of the form.

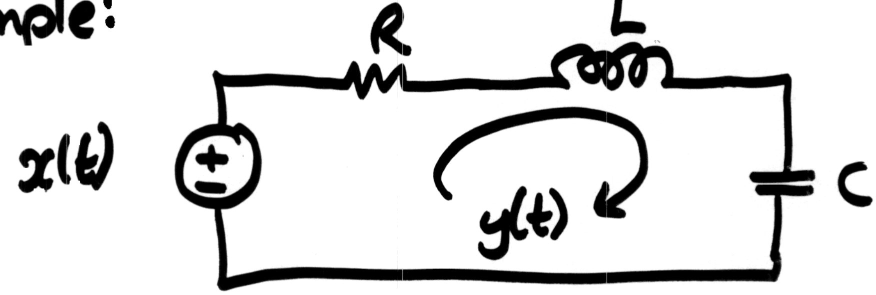
$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t).$$

Note: • only a linear combination of derivatives

• the a_k's and b_k's do not vary with time

• "order" of the equation is N

Example:



System input is a voltage

What is the system output (the loop current)?

$$\frac{1}{C} y(t) + R \frac{d}{dt} y(t) + L \frac{d^2}{dt^2} y(t) = \frac{dx(t)}{dt}$$

Difference Equations

Similar:
$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

Again
- Linear combination of differences
- coefficients are constant
- order is N

To see the differences consider

$$\cancel{\alpha_0 y[n] + \sum_{k=1}^{N} \alpha_k (}$$

$$\alpha_0 y_0[m] + \sum_{k=1}^{N} \alpha_k \left(y[m+k] - y[m+k-1] \right)$$

(True difference equation)

Set $m = n-N$

$$= \alpha_0\, y[n-N] + \sum_{k=1}^{N} \alpha_k \left(y[n\ \ (N-k)] - y[n-(N-k+1)] \right),$$

Set $j = N-k+1.$

$$= \alpha_0\, y[n-N] + \sum_{j=1}^{N} \alpha_{N-j+1} \left(y[n-j+1]\ \ y[n-j] \right)$$

$$= (\alpha_0-\alpha_1)\, y[n-N] + (\alpha_1-\alpha_2)\, y[n-N+1] + \dots + (\alpha_{N-1}-\alpha_N)\, y[n-1] + \alpha_N\, y[n]$$

Solution of Linear Constant Coefficient Differential Equations.

Very similar to the second order case.

just a brief outline.

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = f(t).$$

① Find the natural solution; ie. $y^{(n)}(t)$ such that

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y^{(n)}(t) = 0$$

this by solving the characteristic equation

$$\sum_{k=0}^{N} a_k s^k = 0$$

If the roots r_i are distinct, $y^{(n)}(t)$ has the form

$$y^{(n)}(t) = \sum_{k=0}^{N} c_k e^{r_k t}$$

the i^{th} root is repeated p times we need to consider terms of the form

$$e^{r_i t}, \; t e^{r_i t}, t^2 e^{r_i t}, \dots, t^{p-1} e^{r_i t}$$

(2) Find the forced solution $y^{(f)}(t)$.

This is difficult in general, but if $f(t)$ has a simple form, then so does $y^{(f)}(t)$.

$f(t)$	$y^{(f)}(t)$
	c
e^{-at}	ce^{at}
$\cos(\omega t+\phi)$	$C_1\cos(\omega t)+C_2\sin(\omega t)$

(3) Find $y(t) = y^{(n)}(t) + y^{(f)}(t)$

and resolve constants by applying initial conditions

SOLUTION OF LINEAR CONSTANT COEFFICIENT DIFFERENCE EQUATIONS.

$$\sum_{k=0}^{N} a_k \, y[n-k] = f[n].$$

Approach is entirely analogous

Find natural solution $y^{(n)}[n]$, ie. solve

$$\sum_{n=0}^{N} a_k \, y[n-k] = 0$$

Do this by solving the characteristic equation

$$\sum_{n=0}^{N} a_k \, z^{N-k} = 0$$

If the roots are distinct, and denoted r_i,

$$y^{(n)}[n] = \sum_{i=1}^{N} c_i \, r_i^{\,n}$$

If r_i is repeated p times, we need.

$$r_i^{\,n}, \; n r_i^{\,n}, \; n^2 r_i^{\,n}, \; \ldots, \; n^{p-1} r_i^{\,n}$$

② The forced solution is again difficult in general, but for some simple $f[n]$'s its easy.

$f[n]$	$y^{(f)}[n]$
1	c
α^n	$c\alpha^n$
$\cos(\Omega n+\phi)$	$c_1\cos(\Omega n) + c_2\sin(\Omega n)$

③ Find $y[n] = y^{(n)}[n] + y^{(f)}[n]$

and solve for constants using initial conditions

NOTE THAT THE DIFFERENCE EQUATION CAN BE WRITTEN IN A NICE RECURSIVE FORM:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\Rightarrow \quad y[n] = \underbrace{\frac{1}{a_0} \sum_{k=0}^{M} b_k x[n-k]}_{\text{past and present inputs}} - \underbrace{\frac{1}{a_0} \sum_{k=1}^{N} a_k y[n-k]}_{\text{past outputs.}}$$

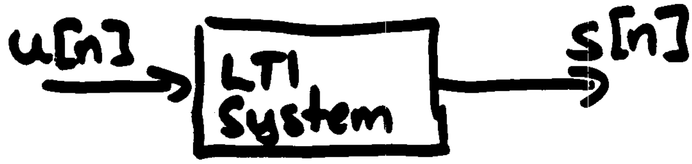
We will need some initial conditions to solve both differential and difference equations

<u>NOTE</u>

- If we have the initial conditions, we can find $y[n]$ numerically using recursion
$$y[n] = \frac{1}{a_0} \sum_{k=0}^{M} b_k x[n-k] - \frac{1}{a_0} \sum_{k=1}^{N} a_k y[n-k]$$
- This gives you an incling of how to solve difficult difference + differential eqns numerical

FINDING THE IMPULSE RESPONSE FROM DIFFERENCE + DIFFERENTIAL EQUATIONS

Consider the step response.



- Since the forcing function is a step, it is often easy to calculate the forced response.
- Furthermore, since the input has been zero for a long time, the initial conditions are zero
- Now what is response to $u[n-1]$?

  Due to ~~a~~ time-invariance, $s[n-1]$

- What is response to $u[n]-u[n-1]$?

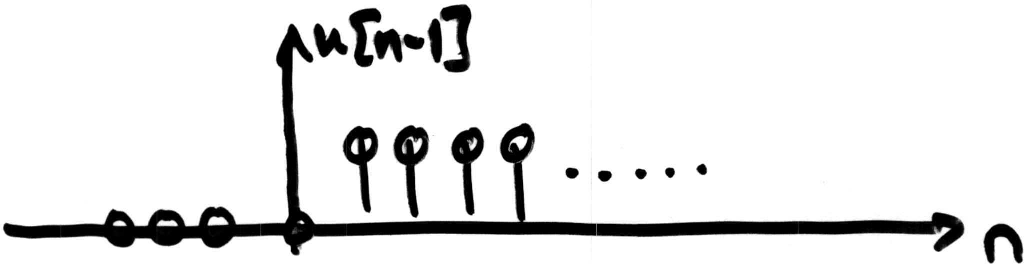
  Due to linearity, $s[n]-s[n-1]$.

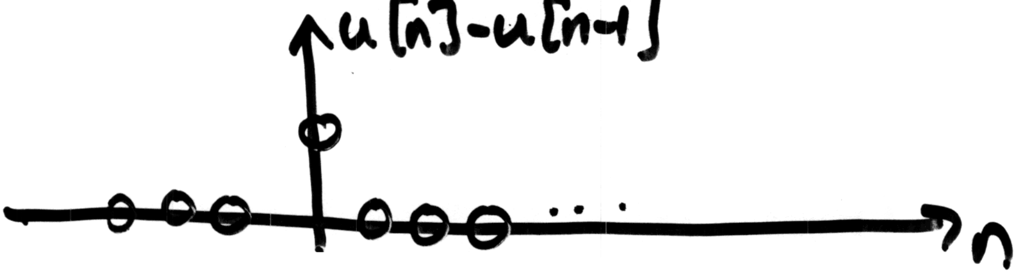
- But what is $u[n]-u[n-1]$? $\delta[n]$!
- Hence $h[n] = s[n]-s[n-1]$
- Does this make sense?

  Previously we found that $s[n] = \sum_{k=-\infty}^{n} h[k]$

  $\Rightarrow s[n]-s[n-1] = \sum_{k=-\infty}^{n} h[k] - \sum_{k=-\infty}^{n-1} h[k] = h[n]$!

u[n]





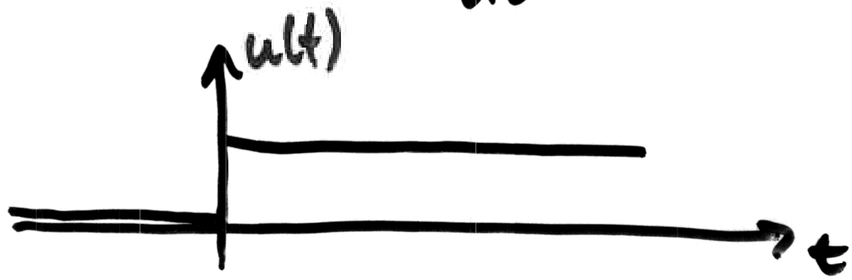
What about the continuous time case

Step response

what is response to $\frac{du(t)}{dt}$?

- Since the system is linear and differentiation is linear, response to $\frac{du(t)}{dt}$ is $\frac{ds(t)}{dt}$ (assuming appropriate existence of derivatives)

- But what is $\frac{du(t)}{dt}$.



In fact $\frac{du(t)}{dt} = \delta(t)$

Hence,

$$\frac{ds(t)}{dt} = h(t) \quad !$$

Does this make sense?

Previously we said that

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

$$A\{h(\tau)\}\Big|_{\tau=t} \qquad A\{h(\tau)\}\Big|_{\tau=-\infty}$$

where A is the "anti derivative operator.

eg. $A\{\tau\} = \frac{\tau^2}{2}$

Now $\frac{ds(t)}{dt} = \underbrace{\frac{d}{dt} A\{h(\tau)\}\Big|_{\tau=t}}_{\text{Derivative and anti-derivative cancel out}}$

0

No dependence on t

$h(t) \quad !$