Optimal Waveform Design For Cognitive Radar

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***Abstract*—A key component of a cognitive radar system is the method by which the transmitted waveform is adapted in response to information regarding the radar environment. The goal of such adaptation methods is to provide a flexible framework that can synthesize waveforms that provide different trade-offs between a variety of performance objectives, and can do so efficiently. In this paper, we propose a waveform design method that efficiently synthesizes waveforms that provide a trade-off between estimation performance for a Gaussian ensemble of targets and detection performance for a specific target. In particular, the method synthesizes (finite length) waveforms that achieve an inherent trade-off between the (Gaussian) mutual information and the signal-to-noise ratio (SNR) for a particular target. In addition, the method can accommodate a variety of constraints on the transmitted spectrum. We show that the waveform design problem can be formulated as a convex optimization problem in the autocorrelation of the waveform, and we develop a customized interior point method for efficiently obtaining a globally optimal waveform.**

***Index Terms*—Waveform design; cognitive radar; convex optimization; spectral factorization; autocorrelation; interior-point method (IPM).**

I. Introduction

Much of the research effort devoted to radar signal processing in the literature has focused on optimizing the design of the receiver [1]. However, with advancements in the fields of digital signal processing [2], neural networks and machine learning [3], and optimization theory [4], and with the emergence of a new discipline called cognitive dynamic systems [5], the stage is set for an examination of the theory and design of cognitive radar systems, in which both the receiver and the transmitter are adapted to the environment. Indeed, there is considerable evidence of such systems in nature, in particular in the echolocation systems of bats and dolphins [6].

To establish what we mean by the term cognitive radar, we will first establish the notion of a cognitive cycle. Figure 1 summarizes the essence of the cognitive cycle in its most basic form. The key aspects are

- Perception of the environment;
- Control exercised on the environment by virtue of feedback of the information that was learnt through perception.

In light of this simplified view of cognition, the notion of a cognitive radar can be established [7] as a complex dynamic system that

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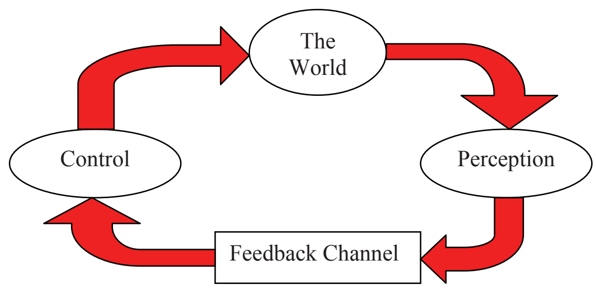

Fig. 1. Cognitive Cycle in its most basic form

- continuously learns about the environment through experience gained from interactions with the environment, and updates the receiver with relevant information on the environment;
- adjusts transmitter's illumination of the environment in an effective and robust manner;
- coordinates the operation of the transmitter and receiver through the use of global feedback.

As suggested by the second item, the development of efficient algorithms for the design (or selection) of the transmitter's waveform is a key enabling step in the construction of a cognitive radar systems. Such algorithms should provide a flexible framework that can synthesize waveforms that provide different trade-offs between a variety of performance objectives. The objectives themselves may also be adapted to the perceived nature of the environment. As we describe in more detail below, the goal of this paper is to develop one such framework, and to illustrate its performance.

A. Problem Statement

The design of radar waveforms has been a topic of considerable research interest for several decades. Generally speaking, the approach to the design of an optimal radar waveform has been task-dependent; e.g., [8]. For example, for the task of detecting a particular target, the output signal-to-noise-ratio (SNR) should be maximized, and the optimal waveform puts all the available energy into the largest mode of the target [8]. For the task of estimating the parameters of a target from a given ensemble, the radar waveform should distribute energy among different modes of the target in such a way as to maximize the mutual information between the received signal and the target ensemble [8]; see also [9].

However, for the purposes of cognitive radar systems a more flexible design framework is required. Rather than optimizing waveforms for a single design criterion, that framework should be able to synthesize waveforms that provide a smooth trade-off between competing design criteria. The development of one such framework is the focus of this paper.

In this paper, we develop an efficient design framework that synthesizes waveforms that provide an optimal trade-off between the detection and estimation criteria described above. The proposed framework can also accommodate a variety of constraints on the transmitted spectrum. In particular, we seek to maximize the mutual information between a Gaussian ensemble of targets and the received signal, subject to a lower bound on the SNR for a specified target, an energy normalization, and bandwidth constraints. Unfortunately, the direct formulation of this problem is not convex, and can be difficult to solve. However, we show that the problem can be transformed into a convex optimization problem in the autocorrelation of the waveform, and we develop a customized interior point algorithm that solves this problem efficiently. The problem transformation was inspired by the successful application of related transformations in the areas of FIR filter design [10], [11], and waveform design for digital communication systems [12], [13], but the objective of the radar waveform design problem is quite different from those considered in that related work.

II. System Model

For simplicity, we will develop our design algorithm for a baseband equivalent model in which all the terms are real-valued. The extension to more general models is straightforward. The waveform transmitted by the radar will be denoted by $x(t)$ and the signal received by the radar can be written as [8]

$$y(t) = \int h(\tau)x(t-\tau)\,d\tau + n(t), \tag{1}$$

where $h(t)$ is the impulse response of the target and $n(t)$ denotes additive Gaussian noise. To simplify the development we will assume that the Gaussian noise has zero mean and is white, with variance σ_n^2, but the extension to colored Gaussian noise is straightforward.

The proposed waveform design algorithm will provide a mechanism for obtaining a smooth trade-off between two competing design criteria, namely the output SNR for a particular target impulse response, and the mutual information between the received signal and a Gaussian ensemble of targets.

To define the target-specific SNR, we first let $\bar{h}(t)$ denote the impulse response of the particular target. If the receiver employs a "target matched" filter (i.e., a filter matched to $\bar{h}(t) * x(t)$, where $*$ denotes convolution), then the target-specific SNR can be written as [14]

$$\mathrm{SNR}_{\bar{h}} = \frac{1}{\sigma_n^2}\int \left|\bar{h}(t) * x(t)\right|^2 dt, \tag{2}$$

where the signal power can be expressed as [15]

$$\int h(\tau_1)\int h(\tau_2)R_x(\tau_2-\tau_1)\,d\tau_2\,d\tau_1, \tag{3}$$

in which $R_x(\tau) = \int x(t)x(t+\tau)\,dt$ is the autocorrelation function of $x(t)$.

To define the mutual information criterion, we let $\breve{h}(t)$ denote a random process that characterizes an ensemble of target impulse responses. Given the transmitted signal, $x(t)$, the (conditional) mutual information between the target and the received signal is [8]

$$I\big(y(t);\breve{h}(t)|x(t)\big). \tag{4}$$

We will consider the case in which the receiver is bandlimited to $[-W/2,\, W/2]$ and has an interval of duration T_y over which the received signal can be observed. In that case, if $\breve{h}(t)$ is a zero-mean Gaussian process, then the (conditional) mutual information can be written as [8]

$$J^{\mathrm{MI}} = T_y\int_{-W/2}^{W/2}\log\big(1+\beta(f)|X(f)|^2\big)\,df, \tag{5}$$

where $X(f)$ is the Fourier Transform of $x(t)$ and

$$\beta(f) = \frac{2S_{\breve{h}}(f)}{\sigma_n^2 T_y}, \tag{6}$$

where $S_{\breve{h}}(f)$ is the power spectral density of the process $\breve{h}(t)$.

For a given specific target response, $\bar{h}(t)$, and a given Gaussian ensemble of targets characterized by $\breve{h}(t)$, our goal is to design the waveform $x(t)$ so as to maximize J^{MI} in (5), subject to $\mathrm{SNR}_{\bar{h}}$ in (2) begin greater than some threshold $\mathrm{SNR}_{\mathrm{Th}}$, and to constraints on the energy and the power spectrum of $x(t)$. In the absence of the SNR constraint and the additional power spectrum (bandwidth) constraints, the optimal solution can be obtained using a "waterfilling" procedure [8], but that procedure is not optimal in the presence of the additional constraints. In order for our design problem to be feasible, $\mathrm{SNR}_{\mathrm{Th}}$ must be less than the maximum achievable $\mathrm{SNR}_{\bar{h}}$ in the presence of the energy and power spectrum constraints on $x(t)$, but as we will mention towards the end of the next section, that quantity can be efficiently computed before the waveform design process is initiated.

In the next section we will show that for a class of waveforms that can be easily implemented, the waveform design problem can be formulated as a convex optimization problem that can be efficiently solved.

III. Convex Design Formulation

We will consider waveforms of the form

$$x(t) = \sum_{\ell=0}^{L-1} g[\ell]p(t-\ell T), \tag{7}$$

where $p(t)$ is a unit-energy waveform, and the parameters $g[\ell]$ are the design variables. Good approximations of such waveforms can be synthesized in a straightforward manner using digital-to-analog conversion of the sequence $g[\ell]$ at a rate $1/T$ using a smoothing filter with impulse response $p(t)$. Typically, T will be chosen to be less than $1/(2W)$, where W is the receiver bandwidth, but that is not necessary. Consistent with the discussion in the previous section, given the specific target response $\bar{h}(t)$ and the ensemble of targets characterized by $\breve{h}(t)$, our goal is to design a sequence $g[\ell]$ of energy E_g that maximizes J^{MI}, subject to $\mathrm{SNR}_{\bar{h}} \geq \mathrm{SNR}_{\mathrm{Th}}$ and the $100\gamma\%$ energy bandwidth of $x(t)$ being at most W. (We will also show how other bounds on the power spectrum of $x(t)$ can be incorporated into the formulation.) Unfortunately, that optimization problem is not convex is $g[\ell]$, and can be quite difficult to solve. In this section we will show how the problem can be reformulated as a convex optimization problem in the autocorrelation sequence of $g[\ell]$, $r_g[m] = \sum_\ell g[\ell]g[\ell+m]$.

To derive that reformulation, we first observe that the (deterministic) autocrorrelation of a waveform of the form in (7) can be written as

$$R_x(\tau) = \int x(t)x(t+\tau)\,dt = \sum_m r_g[m]R_p(\tau - mT). \quad (8)$$

We will normalize the energy of $g[\ell]$ to E_g, i.e.,

$$r_g[0] = E_g, \quad (9)$$

and hence the energy of the transmitted waveform is

$$E_x = R_x(0) = E_g + 2\sum_{m=1}^{L-1} r_g[m]R_p(mT). \quad (10)$$

If the waveform $p(t)$ is self-orthogonal at shifts of integer multiples of T, then $R_p(mT) = \delta_m$, where δ_m is the Kronecker delta, and $E_x = E_g$. Examples of such self-orthogonal waveforms include the rectangular waveforms of duration $\leq T$, the 'sinc' pulse, $\mathrm{sinc}_T(t) = \sin(\pi t/T)/(\pi t/T)$ for $t \neq 0$, and the associated family of pulses with 'root raised cosine' spectra.

The power spectrum of $x(t)$ in (7) can be written as

$$|X(f)|^2 = |G(e^{j2\pi fT})|^2\,|P(f)|^2, \quad (11)$$

where $G(e^{j2\pi F})$ is the Discrete-Time Fourier Transform of $g[\ell]$ and $P(f)$ is the Fourier Transform of $p(t)$. If T is significantly smaller than $1/(2W)$ and if $p(t)$ is a low-pass filter that is close to being self-orthogonal at shifts of integer multiples of T, then $|P(f)|$ can often be approximated as being flat across the band $[-W/2, W/2]$ (see, e.g., [13]), and this approximation will significantly simplify some of the expressions below.

In order to concisely formulate the design problem, we will use the fact that $r_g[0] = E_g$ and the symmetry of $r_g[m]$, and will collect the design variables in the vector

$$\tilde{\mathbf{r}}_g^T = \big[r_g[1],\, r_g[2],\, \ldots,\, r_g[N-1]\big]. \quad (12)$$

Since $|G(e^{j2\pi F})|^2 = E_g + 2\sum_{m=1}^{L-1} r_g[m]\cos(2\pi mF)$, we will find it convenient to define the vector

$$\tilde{\mathbf{v}}(F)^T = 2\big[\cos(2\pi F),\, \cos(2\pi 2F),\, \ldots,\, \cos(2\pi(L-1)F)\big], \quad (13)$$

so that we can write

$$|X(f)|^2 = |P(f)|^2\left(E_g + \tilde{\mathbf{v}}(fT)^T\tilde{\mathbf{r}}_g\right). \quad (14)$$

In order to obtain a convenient expression for the objective of the design problem, we will use the symmetry of the integrand in (5), and will approximate the integral over $[0, W/2]$ by its its K-point Riemann sum. That is, we will maximize

$$\hat{J}_K^{\mathrm{MI}} = 2\Delta_f T_y \sum_{k=0}^{K-1} \log\Big(\alpha(f_k) + \zeta(f_k)\,\tilde{\mathbf{v}}(f_kT)^T\tilde{\mathbf{r}}_g\Big), \quad (15)$$

where $\Delta_f = W/(2K)$, $f_k = k\Delta_f$, $\alpha(f) = 1 + E_g\beta(f)|P(f)|^2$, and $\zeta(f) = \beta(f)|P(f)|^2$. Since the argument of each logarithm in (15) is an affine function of $\tilde{\mathbf{r}}_g$, $\hat{J}_K^{\mathrm{MI}}$ is a concave function of $\tilde{\mathbf{r}}_g$.

Using the expressions in (2) and (3), for waveforms of the form in (7), the SNR of the response to the specified target $\bar{h}(t)$ can be written as

$$\mathrm{SNR}_{\bar{h}} = \sum_{m=-(L-1)}^{L-1} r_g[m]z_m \quad (16)$$

where[1]

$$z_m = \frac{1}{\sigma_n^2}\int R_{\bar{h}}(\lambda)R_p(\lambda - mT)\,d\lambda. \quad (17)$$

Therefore, by defining the elements of the vector $\tilde{\mathbf{w}}$ as $[\tilde{\mathbf{w}}]_m = z_m + z_{-m}$, the constraint that $\mathrm{SNR}_{\bar{h}} \geq \mathrm{SNR}_{\mathrm{Th}}$ can be written as

$$E_g z_0 + \tilde{\mathbf{w}}^T\tilde{\mathbf{r}}_g \geq \mathrm{SNR}_{\mathrm{Th}}, \quad (18)$$

which is an affine constraint in $\tilde{\mathbf{r}}_g$, and hence is convex.

The constraint that $100\gamma\%$ of the energy of the transmitted waveform lies in the band $[-W/2, W/2]$ can be written as $\int_{-W/2}^{W/2}|X(f)|^2\,df \geq \gamma E_x$. Using the above expressions, this constraint can be rewritten as

$$E_g E_{p,W} + 2\sum_{m=1}^{L-1} r_g[m]\int_{-W/2}^{W/2}|P(f)|^2\cos(2\pi mfT)\,df \geq \gamma E_g + 2\gamma\sum_{m=1}^{L-1} r_g[m]R_p(mT), \quad (19)$$

where $E_{p,W} = \int_{-W/2}^{W/2}|P(f)|^2\,df$. This expression can be written more concisely as

$$\tilde{\mathbf{u}}^T\tilde{\mathbf{r}}_g \geq E_g(\gamma - E_{p,W}), \quad (20)$$

where $[\tilde{\mathbf{u}}]_m = 2\int_{-W/2}^{W/2}|P(f)|^2\cos(2\pi mfT)\,df - 2\gamma R_p(mT)$, which is clearly linear in $\tilde{\mathbf{r}}_g$, and hence convex. If $p(t)$ is self-orthogonal at shifts of integer multiples of T and if $P(f)$ can be approximated as being flat across the band $[-W/2, W/2]$, then the expression for the elements of $\tilde{\mathbf{u}}$ can be simplified to $[\tilde{\mathbf{u}}]_m = 2|P(0)|^2\sin(\pi mWT)/(\pi mT)$.

In this paper, we will only enforce spectral constraints of the form in (20), but many more general spectral constraints can be incorporated into our formulation. For instance, using the expression in (14) any constraint of the form $L(f_k) \leq |X(f_k)|^2 \leq U(f_k)$ can be written in terms of affine constraints

[1] If $R_{\bar{h}}(\lambda)$ is approximately constant over the (essential) support of $R_p(\lambda - mT)$, then z_m can be approximated by $|P(0)|^2R_{\bar{h}}(mT)/\sigma_n^2$.

on $\tilde{\mathbf{r}}_g$, and can be easily incorporated into the formulation; see, e.g., [16] for additional discussion.

To complete the formulation, we need to ensure that $r_g[m]$ is a valid autocorrelation sequence. That is, that it can be (spectrally) factorized to obtain a corresponding sequence $g[\ell]$. A necessary and sufficient condition for that to be true is that $R_g(e^{j2\pi F}) = r_g[0] + 2\sum_{m=1}^{L-1} r_g[m]\cos(2\pi mF) \geq 0$ for all $F \in [0, 0.5]$. This semi-infinite constraint can be precisely represented by a (finite) linear matrix inequality (e.g., [16]), but in this paper we will adopt a simple heuristic approach and will approximate the constraint by discretization; e.g., [11]. That is, given an integer N, which is typically of the order of $15L$, we define $F_n = n/(2N)$, $n = 0, 1, \ldots, N$, and enforce

$$R_g(e^{j2\pi F_n}) \geq \epsilon_N, \tag{21}$$

where ϵ_N is a small positive number that provides some control over the behaviour of $R_g(e^{j2\pi F})$ between the discretization points.

Summarizing the above development, the waveform design problem can be written as

$$\begin{aligned}
\min_{\tilde{\mathbf{r}}_g \in \mathbb{R}^{L-1}} \quad & -\sum_{k=0}^{K} \log\big(\alpha(f_k) + \zeta(f_k)\tilde{\mathbf{v}}(f_kT)^T\tilde{\mathbf{r}}_g\big) && \text{(22a)}\\
\text{subject to} \quad & \tilde{\mathbf{w}}^T\tilde{\mathbf{r}}_g \geq \mathrm{SNR}_{\mathrm{Th}} - E_g z_0, && \text{(22b)}\\
& \tilde{\mathbf{u}}^T\tilde{\mathbf{r}}_g \geq E_g(\gamma - E_{p,W}), && \text{(22c)}\\
& \tilde{\mathbf{v}}(F_n)^T\tilde{\mathbf{r}}_g \leq E_g - \epsilon_N, \quad \forall n \in [0, N]. && \text{(22d)}
\end{aligned}$$

Since the objective is a convex function of $\tilde{\mathbf{r}}_g$ and the constraints are linear, this is a convex optimization problem, and an efficient algorithm for finding a globally optimal solution can be developed using interior point methods. We have developed a customized primal potential reduction method (cf. [4]) that employs a damped Newton method with back tracking line search in the re-centering step, and includes an auxiliary phase in which a feasible initial point is found. Once the optimal autocorrelation sequence has been found, an optimal waveform can be constructed by using a standard spectral factorization technique (e.g., [2], [11], [17]) to find a sequence $g[\ell]$ that is a spectral factor of the optimal $r_g[m]$, and then synthesizing the waveform $x(t)$ using (7). As mentioned towards the end of Section II, there is an upper bound on $\mathrm{SNR}_{\mathrm{Th}}$ beyond which the problem in (22) is infeasible. That value is $E_g z_0 + \tilde{\mathbf{w}}^T\tilde{\mathbf{r}}_g^\star$, where $\tilde{\mathbf{r}}_g^\star$ is the optimal solution to the linear program $\max_{\tilde{\mathbf{r}}_g \in \mathbb{R}^{L-1}} \tilde{\mathbf{w}}^T\tilde{\mathbf{r}}_g$, subject to (22c) and (22d).

In the development of the design problem in (22) we considered the case in which there is only one specific target for which an SNR constraint is to be satisfied. However, it is clear from the development that enforcing an SNR constraint for an additional specific target $\bar{h}_i(t)$ simply involves the addition of a single linear constraint of the form $\tilde{\mathbf{w}}_i\tilde{\mathbf{r}}_g \geq \mathrm{SNR}_{\mathrm{Th},i} - E_g z_{0,i}$ to the formulation in (22).

IV. Numerical Examples

In this section we provide some examples of the waveforms that can be obtained using the proposed design method. We consider waveforms $x(t)$ of the form in (7) with $T = 1/2$, $N = 31$, $E_g = 1$, and $p(t)$ being the unit energy rectangular function of duration T. We consider two choices for the specific target waveform, $\bar{h}(t)$, both of which are of the form in (7) with the same parameters as $x(t)$. In the first case, the coefficients are chosen independently from a standard Gaussian distribution and then scaled so that the sequence has unit energy, and in the second the coefficients are samples from a weighted sum of three sinusoids. As befits such a scenario, the observation window of the receiver is $T_y = 2LT$. The receiver bandwidth is $[-1, 1]$, and hence $W = 2$, the energy containment factor, γ, is 0.95, and the power spectral density of the noise is such that the receiver noise variance is 0.1. The power spectral density of the Gaussian target ensemble was chosen to be a windowed, and hence spectrally dispersed, version of $|\bar{H}(f)|^2$.

For the case in which the SNR threshold, $\mathrm{SNR}_{\mathrm{Th}}$, is chosen to be -5 dB, we plot in Figures 2 and 3, the power spectra of the designed waveforms (dashed lines) and compare them with the power spectra of the specific target waveforms, $|\bar{H}(f)|^2$ (solid curves). As can be seen from the figures, the designed waveforms allocate a portion of their energy to the dominant spectral features of the specific target, while also dispersing energy in order to optimize the information obtained about targets from the Gaussian ensemble.

V. Conclusion

In cognitive radar systems, the operation of the transmitter and receiver is coordinated through the use of global feedback. The transmitter should be able to adjust the illumination of the environment in an response to the information obtained form the environment. In this paper, we have considered one such adaptation scheme, and we developed an efficient algorithm for obtaining waveforms of finite (essential) support that provide a smooth trade-off between the SNR of a particular target (or several particular targets) and the mutual information between a Gaussian ensemble of targets and the received signal. The key step in the development of the efficient algorithm was to formulate the design criterion and the constraints as convex functions of the autocorrelation of the parameters that define the waveform. This enabled the development of a customized interior point method for efficiently obtaining an optimal waveform.

In a non-stationary radar environment, the transmitted waveform should be redesigned whenever the specified target or the power spectral density of the Gaussian ensemble change significantly, and in that context the importance of the efficiency with which our formulation can be solved becomes clear. In our development, we assumed the presence of a radar scene analyzer (RSA) [7] that informs the design algorithm of the impulse response of the specified target and the power spectral density of the Gaussian ensemble. In practice, these terms may not be known precisely, and in on-going work we are using insight from [9], [13] to develop design techniques for waveforms that provide robust performance in the presence

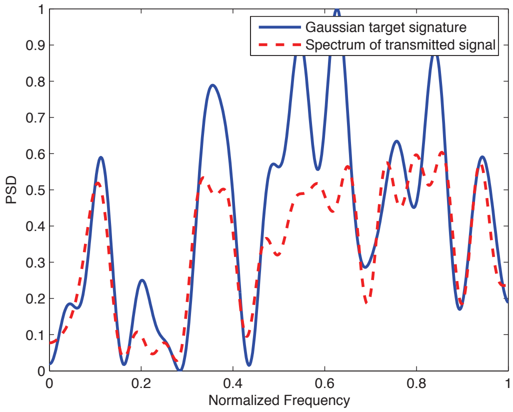

Fig. 2. Power spectrum of the designed radar waveform (dashed) and that of the randomly generated specific target signature (solid).

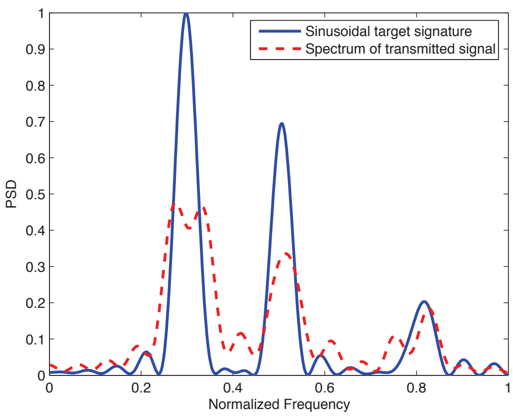

Fig. 3. Power spectrum of the designed radar waveform (dashed) and that of the sinusoidal specific target signature (solid).

of imperfect estimates of the specific target and the power spectral density of the Gaussian ensemble.

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