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A Semidefinite Relaxation Approach to Efficient Soft Demodulation of MIMO 16-QAM

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***Abstract*—Three computationally-efficient list-based soft MIMO demodulators are developed, each of which generates its list using the randomization procedure associated with the semidefinite relaxation (SDR) of a particular hard demodulation problem. The structure of this SDR depends on the signaling scheme, and we will focus on 16-QAM signaling. The key step in the development of the first two demodulators is the derivation of polynomial expressions for the extrinsic information provided by the decoder. These expressions enable this information to be incorporated into the SDR framework. The resulting "List-SDR" demodulators require one semidefinite program (SDP) to be solved at each demodulation-decoding iteration. In the proposed "Single-SDR" demodulator this requirement is reduced to one SDP per channel use by deriving an approximation of the randomization procedure used by the List-SDR demodulator and showing that this approximation enables the decoupling of the processing of the channel measurement from that of the extrinsic information from the decoder. Simulation results show that the proposed demodulators provide considerable reductions in computational cost over several existing soft demodulators, and that these reductions are obtained without incurring a substantial degradation in performance.**

I. Introduction

The combination of bit interleaved coded modulation (BICM, [1]) and iterative demodulation and decoding (IDD, [2]) forms a popular framework for constructing practical multiple-input multiple-output (MIMO) systems that provide good performance at high data rates; e.g., [3]. An important computational bottleneck in this framework is the MIMO soft demodulator, which computes the posterior log likelihood ratio (LLR) of each bit transmitted in a given channel use. One approach to alleviating this bottleneck is to use the "max-log" approximation to approximate the soft demodulation problem of each bit by two hard demodulation problems, and then employ an existing algorithm for hard demodulation; e.g., [4], [5]. An alternative approach is to construct a list of dominant bit-vectors and then approximate the LLRs over that list; e.g., [3], [6]–[9]. (Another alternative is the minimum mean square error soft interference canceler (MMSE-SIC) [10].) In both the "hard" and "list-based" approaches, tree search methods, such as the sphere decoder, have been the basis of most of the proposed algorithms; e.g., [3], [5]–[7]. Although such methods often provide good performance, the distribution of the computational cost is typically rather heavy tailed, which can complicate practical implementation.

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For the hard demodulation approach to soft demodulation, a substantially different algorithm based on semidefinite relaxation (SDR) was proposed in [4]. While that algorithm was developed for QPSK signaling, it has the advantage over tree search methods that its (worst-case) computational cost grows only as a low-order polynomial in the problem size. That said, the algorithm in [4] requires the solution of several semidefinite programs (SDPs) for each demodulation-decoding iteration at each channel use. In [8] it was shown that by adopting the list-based approach to soft demodulation and by exploiting the randomization procedure inherent in the semidefinite relaxation technique, one can construct an SDR-based soft demodulator (the "List-SDR" demodulator) that requires the solution of only one SDP per demodulation-decoding iteration. Furthermore, by approximating that randomization procedure by Bernoulli trials, the number of SDPs to be solved can be further reduced to one per channel use [9], and hence the moniker "Single-SDR" demodulator.

In addition to the fact that the semidefinite relaxation technique was initially developed for quadratic optimization problems with binary variables, the development of the List-SDR [8] and Single-SDR [9] schemes for soft demodulation of QPSK signals was facilitated by the fact that the extrinsic information enters linearly in the *a posteriori* (hard) decision metric. The recent development of different semidefinite relaxation techniques for maximum likelihood (hard) demodulation of 16-QAM symbols [11], [12] provides an opportunity to extend the List-SDR and Single-SDR algorithms to systems that operate at a higher rate. However, this extension is not straightforward, because the extrinsic information does not naturally appear in the low order polynomial form required by the methods in [11], [12]. Furthermore, the simple Bernoulli trials that were used in [9] to approximate the randomization procedure can not be directly applied in the 16-QAM case.

In this paper, we develop two List-SDR soft demodulators for 16-QAM signals, and a Single-SDR soft demodulator. The key step in the development of the List-SDR schemes is the synthesis of a polynomial representation of the extrinsic information that conforms to the SDR scheme in [11], and a polynomial approximation that conforms to the SDR scheme in [12] and hence enables us to take advantage of a customized efficient algorithm that has been developed for the scheme in [12]. The development of the Single-SDR scheme, which only requires one SDP to be solved in each channel use, is based on a new analysis of the distribution of candidate

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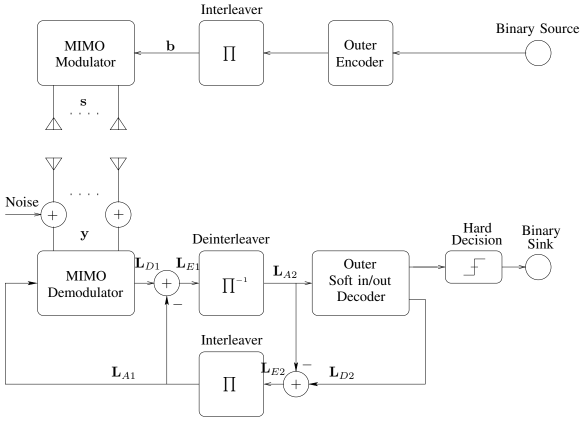

Fig. 1. MIMO BICM-IDD transceiver.

symbols that are generated by the randomization procedures associated with the SDRs in [11] and [12]. As we will show in the simulations section, the proposed Single-SDR demodulator provides performance close to that of existing list sphere decoder and the proposed List-SDR schemes, and better performance than the MMSE-SIC scheme, and that it does so with significantly lower computational cost.

II. System model

We consider the MIMO-BICM-IDD framework in Fig. 1 applied to a narrow-band MIMO system with N_t transmit antennas, N_r receive antennas, and V-BLAST transmission of 16-QAM symbols. (The extension to linear dispersion space-time codes is direct.) The transmitted symbol-vector in Fig. 1 is $\mathbf{s} = \mathcal{M}(\mathbf{b})$, where $\mathcal{M}(\cdot)$ denotes the mapping of a block $\mathbf{b}$ of $4N_t$ interleaved encoded bits to N_t 16-QAM symbols. The received signal vector can be written as $\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{v}$, where the channel matrix $\mathbf{H}$ is assumed to be known at the receiver, and $\mathbf{v}$ denotes the additive white circular complex Gaussian noise, with variance σ^2 per real dimension. By defining $\tilde{\mathbf{s}} = [\mathrm{Re}\{\mathbf{s}^T\}, \mathrm{Im}\{\mathbf{s}^T\}]^T$ and defining $\tilde{\mathbf{y}}$ and $\tilde{\mathbf{v}}$ in an analogous way, we can construct the following real-valued representation of the received signal,

$$\tilde{\mathbf{y}} = \tilde{\mathbf{H}}\tilde{\mathbf{s}} + \tilde{\mathbf{v}}. \tag{1}$$

The role of the soft demodulator in Fig. 1 is to compute the posterior log likelihood-ratio (LLR) of each interleaved encoded bit. These LLRs can be written as (e.g., [3])

$$L_i = \log \frac{\Sigma_{\mathcal{L}_{i,+1}} \exp(-\breve{D}(\tilde{\mathbf{s}})/(2\sigma^2))}{\Sigma_{\mathcal{L}_{i,-1}} \exp(-\breve{D}(\tilde{\mathbf{s}})/(2\sigma^2))}, \tag{2}$$

where $\tilde{\mathbf{s}} = \tilde{\mathcal{M}}(\mathbf{b})$, $\mathcal{L} = \{\mathbf{b} \in \{\pm 1\}^{4N_t}\}$ denotes the list of all bit-vectors, $\mathcal{L}_{i,\pm 1} = \{\mathbf{b} \in \mathcal{L} | b_i = \pm 1\}$, and $\breve{D}(\tilde{\mathbf{s}}) \triangleq \|\tilde{\mathbf{y}} - \tilde{\mathbf{H}}\tilde{\mathbf{s}}\|_2^2 - 2\sigma^2 \log p(\tilde{\mathbf{s}})$, where $p(\tilde{\mathbf{s}})$ represents the extrinsic information. In an IDD scheme, $p(\tilde{\mathbf{s}})$ is not available and is usually approximated by assuming the elements of $\mathbf{b}$ to be independent and by using the extrinsic information from the previous iteration of the decoder to approximate $p_i(\tilde{s}_i)$; i.e., $p(\tilde{\mathbf{s}}) \simeq \prod_{i=1}^{2N_t} \hat{p}_i(\tilde{s}_i)$, where $\tilde{s}_i$ is the ith element of $\tilde{\mathbf{s}}$. Hence the soft demodulator makes the approximation (e.g., [3])

$$\breve{D}(\tilde{\mathbf{s}}) \simeq D(\tilde{\mathbf{s}}) \triangleq \|\tilde{\mathbf{y}} - \tilde{\mathbf{H}}\tilde{\mathbf{s}}\|_2^2 - 2\sigma^2 \sum_{i=1}^{2N_t} \log \hat{p}_i(\tilde{s}_i). \tag{3}$$

One common approach to reducing the computational cost of (2) is to apply the "max-log" approximation

$$L_i \simeq \frac{1}{2\sigma^2}\Big(\min_{\mathbf{b}\in\hat{\mathcal{L}}_{i,-1}} D(\tilde{\mathbf{s}}) - \min_{\mathbf{b}\in\hat{\mathcal{L}}_{i,+1}} D(\tilde{\mathbf{s}}) \Big), \tag{4}$$

over a list $\hat{\mathcal{L}} \subseteq \mathcal{L}$. The approximate LLR in (4) can be generated in two ways. The first is based on selecting $\hat{\mathcal{L}} = \mathcal{L}$ and solving the minimization problems in (4) using the direct application of "hard" demodulation techniques; e.g., [4], [5]. The second is based on efficiently selecting a list $\hat{\mathcal{L}}$ of bit-vectors with small values of $D(\tilde{\mathbf{s}})$ and then performing an exhaustive search over $\hat{\mathcal{L}}_{i,\pm 1}$ to solve the problems in (4), (e.g., [3]). (The list-based approach can also employ other approximations of the LLRs; e.g., [7].) The demodulators proposed herein are of the second type, in the sense that they (implicitly) generate a list of candidate bit-vectors via a semidefinite relaxation (SDR) of the hard demodulation problem for 16-QAM signals. Two relaxations to that problem are described in the next section. The proposed soft demodulators will be introduced in Sections IV and V.

III. ML demodulation of MIMO 16-QAM via SDR

Maximum likelihood (hard) demodulation for the MIMO system in (1) with 16-QAM signaling can be expressed as the following (NP-hard) optimization problem

$$\min_{\tilde{\mathbf{s}}\in\{\pm 1,\pm 3\}^{2N_t}} \|\tilde{\mathbf{y}} - \tilde{\mathbf{H}}\tilde{\mathbf{s}}\|_2^2. \tag{5}$$

Now let us define

$$\mathbf{x} \triangleq \begin{bmatrix} \breve{\mathbf{s}} \\ c \end{bmatrix}, \quad \breve{\mathbf{s}} \triangleq c\tilde{\mathbf{s}}, \quad \tilde{\mathbf{Q}} \triangleq \begin{bmatrix} \tilde{\mathbf{H}}^T\tilde{\mathbf{H}} & -\tilde{\mathbf{H}}^T\tilde{\mathbf{y}} \\ -\tilde{\mathbf{y}}^T\tilde{\mathbf{H}} & 0 \end{bmatrix}, \tag{6}$$

where $c \in \{\pm 1\}$. By observing that $\mathbf{x}^T\tilde{\mathbf{Q}}\mathbf{x}$ is within a constant of $\|\tilde{\mathbf{y}} - \tilde{\mathbf{H}}\tilde{\mathbf{s}}\|_2^2$, and by denoting $\mathbf{X} = \mathbf{x}\mathbf{x}^T$, the problem in (5) can be shown (e.g., [11], [12]) to be equivalent to

$$\min_{\mathbf{X}\succeq 0} \mathrm{Trace}(\mathbf{X}\tilde{\mathbf{Q}}) \tag{7a}$$
$$\text{s.t.} \quad [\mathbf{X}]_{ii} \in \mathcal{B}, \quad i = 1, \dots, 2N_t \tag{7b}$$
$$[\mathbf{X}]_{ii} = 1, \quad i = 2N_t + 1, \tag{7c}$$
$$\mathrm{rank}(\mathbf{X}) = 1, \tag{7d}$$

where $\mathcal{B} = \{L, U\}$, $L = +1, U = +9$, and $\mathbf{X} \succeq 0$ means that $\mathbf{X}$ is positive semidefinite. This problem is still an NP-hard problem due to the constraints in (7b) and (7d).

The general philosophy of the semidefinite relaxation approach is to "relax" (7) by removing the rank-1 constraint, solve the resulting problem for $\mathbf{X}$, and then use $\mathbf{X}$ to generate candidates for the solution to (5), usually via a randomization procedure. In the case of binary signaling, the related optimization problem is an SDP and an efficient algorithm for that class of SDP was developed in [13]. However, in the case of

16-QAM the relaxed problem still appears to be hard to solve, due to the constraint in (7b). Two methods for dealing with that constraint were proposed in [11] and [12]. The method in [11] precisely transforms the constraint in (7b) into linear constraints in a higher dimensional space, and hence will be called the "increased-dimension" relaxation. In contrast, the method in [12] simply relaxes the constraint $[\mathbf{X}]_{ii} \in \mathcal{B}$ to the linear constraints $L \leq [\mathbf{X}]_{ii} \leq U$, and hence the dimension of the variable is the same as that in (7). (We will call this method the "fixed-dimension" relaxation.) For completeness, in the following sections we will state the SDPs solved in these methods, and the randomization procedure. The details can be found in [11] and [12].

A. Increased-dimension relaxation [11]

First, we define

$$\mathbf{W} \triangleq \begin{bmatrix} \mathbf{W}_{11} & \mathbf{w}_{12} & \mathbf{W}_{13} \\ \mathbf{w}_{21}^T & w_{22} & \mathbf{w}_{23}^T \\ \mathbf{W}_{31} & \mathbf{w}_{32} & \mathbf{W}_{33} \end{bmatrix}, \tag{8}$$

where each $\mathbf{W}_{ij}$ is of size $2N_t \times 2N_t$, and w_{22} is a scalar. The semidefinite relaxation of (7) obtained in [11] is

$$\min_{\mathbf{W} \succeq 0} \ \mathrm{Trace}(\mathbf{WQ}) \tag{9a}$$

$$\text{s.t.} \quad \mathrm{diag}(\mathbf{W}_{11}) - \mathbf{w}_{32} = \mathbf{0}, \qquad w_{22} = 1, \tag{9b}$$

$$\mathrm{diag}(\mathbf{W}_{33}) - (U+L)\mathrm{diag}(\mathbf{W}_{11}) + UL\mathbf{1} = \mathbf{0}, \tag{9c}$$

where the operator $\mathrm{diag}(\cdot)$ constructs a vector of the diagonal elements of its matrix argument, $\mathbf{1}$ is a vector with all its elements equal to 1, and

$$\mathbf{Q} \triangleq \begin{bmatrix} \tilde{\mathbf{Q}}_{(2N_t+1)\times(2N_t+1)} & \mathbf{0}_{(2N_t+1)\times 2N_t} \\ \mathbf{0}_{2N_t\times(2N_t+1)} & \mathbf{0}_{2N_t\times 2N_t} \end{bmatrix}, \tag{10}$$

in which $\tilde{\mathbf{Q}}$ was defined in (6). The computational cost of solving this SDP using general purpose interior point methods is $O(N_t^{6.5}\log\epsilon^{-1})$, where ϵ is the solution accuracy.

An approximate solution to (5) can be obtained by performing a randomization procedure based on the following sub-matrix of $\mathbf{W}_{\text{opt}}$, the solution of (9):

$$\tilde{\mathbf{W}}_{\text{opt}} \triangleq \begin{bmatrix} \mathbf{W}_{\text{opt},11} & \mathbf{w}_{\text{opt},12} \\ \mathbf{w}_{\text{opt},21} & w_{\text{opt},22} \end{bmatrix}, \tag{11}$$

where the structure is conformal with (8). Having computed the (Cholesky) factorization $\tilde{\mathbf{W}}_{\text{opt}} = \mathbf{V}^T\mathbf{V}$, the procedure chooses a random vector $\mathbf{u}$ from the uniform distribution on the unit sphere and computes $\breve{\mathbf{s}}' = \mathcal{Q}\big(\frac{\mathbf{V}^T\mathbf{u}}{\mathbf{v}_{2N_t+1}^T\mathbf{u}}\big)$, where $\mathcal{Q}(\cdot)$ is a quantizer to the values in $\mathcal{A} = \{\pm 1, \pm 3\}$. This randomization step is then repeated, and among all generated symbol-vectors, we pick the $\tilde{\mathbf{s}}' \triangleq [\breve{s}_1', \ldots, \breve{s}_{2N_t}']^T$ with the largest likelihood.

B. Fixed-dimension relaxation [12]

The semidefinite program solved in the method in [12] is

$$\min_{\mathbf{X} \succeq 0} \ \mathrm{Trace}(\mathbf{X}\tilde{\mathbf{Q}}) \tag{12a}$$

$$\text{s.t.} \quad L \leq [\mathbf{X}]_{ii} \leq U, \quad i = 1, \ldots, 2N_t \tag{12b}$$

$$[\mathbf{X}]_{ii} = 1, \quad i = 2N_t + 1. \tag{12c}$$

A fast interior point algorithm for a general class of SDPs that includes (12) was proposed in [13]. Based on that algorithm, a specialized algorithm that exploits the structure in (12) was proposed concurrently in [14] and [15]. That algorithm has a computational cost of $O(N_t^{3.5}\log\epsilon^{-1})$. The randomization procedure for this scheme is analogous to that in [11], but is based on $\mathbf{X}_{\text{opt}}$.

Interestingly, it was concurrently shown in [14] and [15] that the optimal values of the semidefinite programs in (9) and (12) are equal. It was also proved in [14] that the optimum solutions to problems (9) (i.e., (11)) and (12) are equal.

IV. List-SDR Soft MIMO 16-QAM demodulator

The principle behind the List-SDR demodulator for QPSK signaling [8] is to construct the demodulation list from the bit-vectors generated by the randomization procedure associated with the semidefinite relaxation of the approximate maximum *a posteriori* detection problem,

$$\min_{\tilde{\mathbf{s}} \in \mathcal{A}^{2N_t}} D(\tilde{\mathbf{s}}), \tag{13}$$

where $\mathcal{A}$ contains the points on one dimension of the square signaling alphabet and $D(\tilde{\mathbf{s}})$ was defined in (3). In the QPSK case, that SDR can be constructed in a straightforward manner because the standard SDR technique can be applied to any quadratic objective and the extrinsic information enters linearly in $D(\tilde{\mathbf{s}})$. However, in the case of 16-QAM, this information enters in a non-polynomial fashion. The goal of this section is to derive a precise third order polynomial expression for the extrinsic information that can be represented in the form of (9a) and hence leads to an "increased-dimension" List-SDR demodulator, and a second order approximation of the extrinsic information that can be represented in the form of (12a) and hence leads to a fixed-dimension List-SDR demodulator.

To obtain the precise third order polynomial expression for the extrinsic information, for each $i \in 1, \ldots, 2N_t$ we need to determine a quadruple (a_i, b_i, c_i, d_i) such that $\log\hat{p}_i(\tilde{s}_i) = a_i\tilde{s}_i^3 + b_i\tilde{s}_i^2 + c_i\tilde{s}_i + d_i$ for all $\tilde{s}_i \in \mathcal{A}$, where $\mathcal{A} = \{\pm 1, \pm 3\}$. If we define $\bar{s}_1 = -3$, $\bar{s}_2 = -1$, $\bar{s}_3 = +1$ and $\bar{s}_4 = +3$, this corresponds to solving

$$\underbrace{\begin{bmatrix} \bar{s}_1^3 & \bar{s}_1^2 & \bar{s}_1 & 1 \\ \bar{s}_2^3 & \bar{s}_2^2 & \bar{s}_2 & 1 \\ \bar{s}_3^3 & \bar{s}_3^2 & \bar{s}_3 & 1 \\ \bar{s}_4^3 & \bar{s}_4^2 & \bar{s}_4 & 1 \end{bmatrix}}_{\triangleq \mathbf{C}_{\text{3rd}}} \begin{bmatrix} a_i \\ b_i \\ c_i \\ d_i \end{bmatrix} = \begin{bmatrix} \log p_i(\tilde{s}_i = \bar{s}_1) \\ \log p_i(\tilde{s}_i = \bar{s}_2) \\ \log p_i(\tilde{s}_i = \bar{s}_3) \\ \log p_i(\tilde{s}_i = \bar{s}_4) \end{bmatrix}. \tag{14}$$

Since $\mathbf{C}_{\text{3rd}}$ is non-singular, this system has a unique solution. Collecting these solutions in the vectors $\mathbf{a}$,$\mathbf{b}$,$\mathbf{c}$ and $\mathbf{d}$, we obtain the following third-order polynomial expression for $D(\tilde{\mathbf{s}})$ in (3)

$$D(\tilde{\mathbf{s}}) = \|\tilde{\mathbf{y}} - \tilde{\mathbf{H}}\tilde{\mathbf{s}}\|_2^2 - 2\sigma^2(\mathbf{a}^T\tilde{\mathbf{s}}^3 + \mathbf{b}^T\tilde{\mathbf{s}}^2 + \mathbf{c}^T\tilde{\mathbf{s}} + \mathbf{d}^T\mathbf{1}), \tag{15}$$

where $\tilde{\mathbf{s}}^n$ is defined element-wise. Given this expression for $D(\tilde{\mathbf{s}})$, the "increased-dimension" SDR of (13) takes the form of the SDP in (9), but with the matrices $\tilde{\mathbf{Q}}$ in (6) and $\mathbf{Q}$ in (10) being replaced by

$$\tilde{\mathbf{Q}} = \begin{bmatrix} \tilde{\mathbf{H}}^T\tilde{\mathbf{H}} - 2\sigma^2\mathrm{Diag}(\mathbf{b}) & -\tilde{\mathbf{H}}^T\tilde{\mathbf{y}} - \sigma^2\mathbf{c} \\ -\tilde{\mathbf{y}}^T\tilde{\mathbf{H}} - \sigma^2\mathbf{c}^T & 0 \end{bmatrix} \tag{16}$$

where the operator $\mathrm{Diag}(\cdot)$ constructs a diagonal matrix with its argument on the diagonal, and

$$\mathbf{Q} = \begin{bmatrix} \tilde{\mathbf{Q}}_{2N_t+1\times 2N_t+1} & -\sigma^2\mathrm{Diag}(\mathbf{a}) \\ -\sigma^2\mathrm{Diag}(\mathbf{a}) & \mathbf{0}_{2N_t\times 2N_t} \end{bmatrix}. \tag{17}$$

To apply the "fixed-dimension" SDR technique in the List-SDR framework, we need a quadratic expression for the extrinsic information. Since $\tilde{s}_i$ takes one of four possible values, $\log \hat{p}_i(s_i)$ can not be interpolated by a quadratic. Instead, we will choose the quadratic that minimizes the squared error between $\log \hat{p}_i(s_i)$ and $b_i\tilde{s}_i^2 + c_i\tilde{s}_i + d_i$ over $\tilde{s}_i \in \mathcal{A}$. That is, we will solve

$$\underbrace{\begin{bmatrix} \sum \bar{s}^4 & \sum \bar{s}^3 & \sum \bar{s}^2 \\ \sum \bar{s}^3 & \sum \bar{s}^2 & \sum \bar{s} \\ \sum \bar{s}^2 & \sum \bar{s} & 4 \end{bmatrix}}_{\triangleq \mathbf{C}_{\text{2nd}}} \begin{bmatrix} b_i \\ c_i \\ d_i \end{bmatrix} = \begin{bmatrix} \sum \bar{s}^2 \log p_i(\tilde{s}_i = \bar{s}) \\ \sum \bar{s} \log p_i(\tilde{s}_i = \bar{s}) \\ \sum \log p_i(\tilde{s}_i = \bar{s}) \end{bmatrix}, \tag{18}$$

where all the summations are over all $\bar{s} \in \mathcal{A}$. By collecting the solution for each $i \in \{1, \ldots, 2N_t\}$ in the vectors $\mathbf{b}$,$\mathbf{c}$ and $\mathbf{d}$, the second-order approximation of (3) can be written as

$$D(\tilde{\mathbf{s}}) \simeq \|\tilde{\mathbf{y}} - \tilde{\mathbf{H}}\tilde{\mathbf{s}}\|_2^2 - 2\sigma^2(\mathbf{b}^T\tilde{\mathbf{s}}^2 + \mathbf{c}^T\tilde{\mathbf{s}} + \mathbf{d}^T\mathbf{1}). \tag{19}$$

Therefore, the "fixed-dimension" SDR of (13) takes the form of the SDP in (12), but with $\tilde{\mathbf{Q}}$ in (6) replaced by

$$\tilde{\mathbf{Q}} = \begin{bmatrix} \tilde{\mathbf{H}}^T\tilde{\mathbf{H}} - 2\sigma^2\mathrm{Diag}(\mathbf{b}) & -\tilde{\mathbf{H}}^T\tilde{\mathbf{y}} - \sigma^2\mathbf{c} \\ -\tilde{\mathbf{y}}^T\tilde{\mathbf{H}} - \sigma^2\mathbf{c}^T & 0 \end{bmatrix}. \tag{20}$$

Now that we have obtained the polynomial expressions in (15) and (19) we now briefly outline the two List-SDR algorithms: At each demodulation iteration, the first step is to solve either[1] (14) or (18), and then solve the corresponding increased-dimension or fixed-dimension SDP. The randomization procedure is then used to generate a preliminary list of candidate vectors $\hat{\mathcal{L}}'$. The list $\hat{\mathcal{L}}$ that is used in soft demodulation approximation (4), is then constructed by adding to $\hat{\mathcal{L}}'$ the single bit-flippings of selected best K bit-vectors in $\hat{\mathcal{L}}'$. These steps have been summarized in Tab. I.

Although the increased-dimension and fixed-dimension approaches are equivalent in the absence of extrinsic information [14], [15], it is clear from the above derivation that the fixed-dimension SDR requires an approximation of the extrinsic information. However, as we will show in our simulations, good performance can be obtained using the fixed-dimension SDR, and that approach has the advantage that the resulting SDP is approximately half the size and has a structure that enables the application of the computationally-efficient interior point algorithm in [14], [15]; see also Tab. III.

[1]Since $\mathbf{C}_{\text{2nd}}$ and $\mathbf{C}_{\text{3rd}}$ are not dependent on the channel information nor the extrinsic information, their LU factorizations can be pre-computed.

TABLE I
LIST-SDR ALGORITHM

- Data: $\mathbf{X}_{\text{opt}}$ the solution to (12) (or $\tilde{\mathbf{W}}_{\text{opt}}$ if using (9))
- Parameters: M, the number of randomization iterations; K, the maximum number of best list members to perform bit-flipping on.
- Output: $\hat{\mathcal{L}}$, the enriched list.

1. Initialize $\hat{\mathcal{L}}$ and $\hat{\mathcal{L}}'$ empty and $m = 0$.
2. Compute a (Cholesky) factor $\mathbf{V}$ of $\mathbf{X}_{\text{opt}}$ such that $\mathbf{X}_{\text{opt}} = \mathbf{V}^T\mathbf{V}$.
3. Choose $\mathbf{u}$ from the uniform distribution on the unit sphere.
4. Construct $\breve{\mathbf{s}} = \mathcal{Q}\Big(\frac{\mathbf{V}^T\mathbf{u}}{\mathbf{v}_{2N_t+1}^T\mathbf{u}}\Big)$, $\tilde{\mathbf{s}} = [\breve{s}_1, \ldots, \breve{s}_{2N_t}]^T$ and increment m.
5. Find the bit-vector $\mathbf{b}$ corresponding to $\tilde{\mathbf{s}}$.
6. If $\mathbf{b}$ is not in $\hat{\mathcal{L}}'$ add it to $\hat{\mathcal{L}}'$.
7. If $m < M$ return to 3.
8. Add all the bit-vectors in $\hat{\mathcal{L}}'$ and the bit-flippings of its K best bit-vectors to $\hat{\mathcal{L}}$ and return $\hat{\mathcal{L}}$.

V. SINGLE-SDR SOFT MIMO 16-QAM DEMODULATOR

The principle that underlies the Single-SDR technique for QPSK signaling [9], is to approximate the randomization procedure in the List-SDR scheme by independent (scalar) Bernoulli trials. This provides an opportunity to decouple the processing of the channel measurement from that of the extrinsic information, and hence reduces the number of SDPs that need to be solved from one per demodulation-decoding iteration to one per channel use. In order to extend this idea to the case of 16-QAM signaling, we will first derive an analytic expression for the probability that each element of the candidate symbol-vector $\tilde{\mathbf{s}}$ generated by randomization procedure in Section IV takes each of the values in $\mathcal{A} = \{\pm 1, \pm 3\}$.

For each element $\tilde{s}_i$ of $\tilde{\mathbf{s}}$ generated by the randomization procedure, we define $p_i^r(\tilde{s}_i)$ to be the probability that $\tilde{s}_i$ takes each of the values in $\mathcal{A} = \{\pm 1, \pm 3\}$. (We have added the superscript r to $p_i^r(\tilde{s}_i)$ to distinguish between the randomization probability and the approximate *a priori* probability $\hat{p}_i(\tilde{s}_i)$ provided by the decoder.) Since $\mathbf{u}$ is uniformly distributed on the unit sphere, we can compute $p_i^r(\tilde{s}_i)$ by evaluating the probability of $\frac{\mathbf{v}_i^T\mathbf{u}}{\mathbf{v}_{2N_t+1}^T\mathbf{u}}$ being in the corresponding interval for the set $\{(-\infty, -2], [-2, 0], [0, +2], [+2, +\infty)\}$, where $\mathbf{v}_i$ is the ith column of the Cholesky factor $\mathbf{V}$; see [14]. If we define

$$\alpha_i = \tan^{-1}\Big(\frac{2 - \|\mathbf{v}_i\|_2\cos\theta_i}{\|\mathbf{v}_i\|_2\sin\theta_i}\Big), \tag{21a}$$

$$\beta_i = \tan^{-1}\Big(\frac{2 + \|\mathbf{v}_i\|_2\cos\theta_i}{\|\mathbf{v}_i\|_2\sin\theta_i}\Big), \tag{21b}$$

the probabilities that a given symbol will be generated by the randomization procedure in Section IV can be written as

$$p_i^r(\tilde{s}_i = -3) = (\pi/2 - \beta_i)/\pi, \tag{22a}$$
$$p_i^r(\tilde{s}_i = -1) = (\theta_i - \pi/2 + \beta_i)/\pi, \tag{22b}$$
$$p_i^r(\tilde{s}_i = +1) = (\alpha_i + \pi/2 - \theta_i)/\pi, \tag{22c}$$
$$p_i^r(\tilde{s}_i = +3) = (\pi/2 - \alpha_i)/\pi. \tag{22d}$$

Hence, by assuming independence between each element of $\tilde{\mathbf{s}}$, we can approximate the randomization procedure by generating the candidate symbol-vectors using independent discrete random number generators with the probabilities computed in (22). One advantage of this approach is that it avoids the

TABLE II
SINGLE-SDR ALGORITHM

- Data: $p_i^r(\tilde{s}_i)$, $i = 1, \ldots, 2N_t$ computed in (22), $\hat{p}_i(\tilde{s}_i)$, $i = 1, \ldots, 2N_t$ computed using the extrinsic information from decoder.
- Parameters: M, the number of randomization iterations; K, the maximum number of best list members to perform bit-flipping on.
- Output: $\hat{\mathcal{L}}$, the enriched list.

1. Initialize $\hat{\mathcal{L}}$ and $\hat{\mathcal{L}}'$ empty and $m = 0$.
2. Compute $\breve{p}_i^r(\tilde{s}_i)$, $i = 1 \ldots, 2N_t$ as in (23).
3. Generate each $\tilde{s}_i$ independently according to the probability distributions with probabilities computed in step 2, and increment m.
4. Find the bit-vector $\mathbf{b}$ corresponding to $\tilde{\mathbf{s}}$.
5. If $\mathbf{b}$ is not in $\hat{\mathcal{L}}'$ add it to $\hat{\mathcal{L}}'$.
6. If $m < M$ return to 3.
7. Add all the bit-vectors in $\hat{\mathcal{L}}'$ and the bit-flippings of its K best bit-vectors to $\hat{\mathcal{L}}$ and return $\hat{\mathcal{L}}$.

TABLE III
DOMINANT COMPUTATIONAL COSTS

Demodulator	Dominant Computational Cost
List-SDR (increased-dimension)	$O(TN_t^{6.5})$
List-SDR (fixed-dimension)	$O(TN_t^{3.5})$
Single-SDR (fixed-dimension)	$O(N_t^{3.5})$
MMSE-SIC	$O(TN_t^{4})$

computation of $\mathbf{V}^T\mathbf{u}$ in each randomization iteration. More importantly, as we will show below, it provides the opportunity to decouple the processing of the channel measurement from that of the extrinsic information, and hence we need to solve only one SDP per channel use.

At the second and subsequent iterations, the distributions in (22) should be modified to incorporate the extrinsic information provided by the decoder. Since this extrinsic information is, by construction, independent from the information provided by the channel [2], the modified distributions for the randomized demodulator can be chosen to be

$$\breve{p}_i^r(\tilde{s}_i = \bar{s}) = \kappa_i p_i^r(\tilde{s}_i = \bar{s})\hat{p}_i(\tilde{s}_i = \bar{s}), \tag{23}$$

where $\bar{s} \in \mathcal{A}$, $\hat{p}_i(\tilde{s}_i)$ was defined just before (3), and κ_i is a normalization constant such that $\sum_{\bar{s}\in\mathcal{A}} \breve{p}_i^r(\tilde{s}_i = \bar{s}) = 1$. After generating the preliminary list $\hat{\mathcal{L}}'$ using (23), the generation of the final list $\hat{\mathcal{L}}$ is the same as that described in Section IV. In the Single-SDR scheme, the SDP is solved only in the first demodulation-decoding iteration where no extrinsic information is available. Therefore, the objective is quadratic and hence we can use the fixed-dimension SDR approach in (12) directly. For convenience, the Single-SDR algorithm is presented in Tab. II.

We have summarized the dominant component of the computational cost per channel use of each of the proposed demodulators in Tab. III, in which T is the number of demodulation-decoding iterations. That table also includes the computational cost of the MMSE-SIC demodulator in [10].

VI. SIMULATION RESULTS

We will consider a MIMO independent Rayleigh block fading channel with $N_t = N_r = 4$. The outer code is the rate $1/2$ punctured turbo code of (input) block length $8{,}192$ that was used in [3].[2] At the receiver, four demodulation-decoding iterations are performed, and in each demodulation iteration eight iterations of BCJR decoding of the constituent codes are performed. We will consider six soft demodulators: the two List-SDR demodulators and the Single-SDR demodulator proposed herein, the LISS demodulator [6], the list sphere decoder [3], and the MMSE-SIC demodulator (e.g., [10]). For each demodulator the LLRs were clipped to $[-5, +5]$. In the Single-SDR scheme and the 'fixed-dimension' List-SDR scheme, the specialized interior point algorithm developed in [14], [15] was used to solve the SDPs, and in the 'increased-dimension' List-SDR scheme SeDuMi [16] was used. In all three cases the SDPs were solved to an accuracy of $\epsilon = 10^{-1}$, $M = 50$ randomizations were performed, and the single bit flippings of the $K = 20$ best list members were added to the list. For the LISS demodulator we considered a stack size of $S = 500$, a list size of $L = 80$ and the list augmentation scheme in [6], and for the list sphere decoder the list size was set to $L = 512$.

Fig. 2 compares the BER performance of these demodulators. For reference, the SNR threshold of rate $1/2$ coded 16-QAM is approximately 6.9 dB; cf. [3]. From Fig. 2 it is apparent that because it can incorporate the extrinsic information without approximation, the performance of the 'increased-dimension' List-SDR demodulator is better than that of the 'fixed-dimension' List-SDR scheme. The performance of both these demodulators is better than that of the list sphere decoder and the MMSE-SIC demodulator and is close to that of the LISS demodulator. It is also apparent from Fig. 2 that the Single-SDR demodulator provides better performance than the MMSE-SIC demodulator, and that by increasing the number of randomization iterations it can achieve performance close to that of the other schemes.

In order to show that the proposed demodulators achieve this performance at a low computational cost, we explicitly counted the number of floating point operations (FLOPs) required to generate the list in each scheme and also the number of FLOPs required to compute the metrics. In Fig. 3 we plot the average number of FLOPs per channel use, and this quantifies the computational advantage of the List-SDR and Single-SDR demodulators over the list sphere decoder and the LISS demodulator.[3] It also quantifies the computational advantages of the Single-SDR scheme over the MMSE-SIC demodulator for different numbers of randomization iterations, M. Furthermore, the computational cost distribution of the List-SDR and Single-SDR schemes is concentrated around the mean, whereas the distributions of the list sphere decoder and the LISS demodulator have quite long tails. To illustrate that fact, we have plotted in Fig. 4 the empirical probability density of the computational cost per channel use at an SNR of 9.75 dB. Fig. 4 also illustrates that most of the computational cost of the Single-SDR demodulator is almost always less than

[2]Some results for a short convolutional outer code appear in [14].

[3]Since we use a precompiled package for the increased-dimension List-SDR demodulator, it has been omitted from Fig. 3. However, Tab. III suggests that it would be substantially more expensive than the schemes considered in Fig. 3.

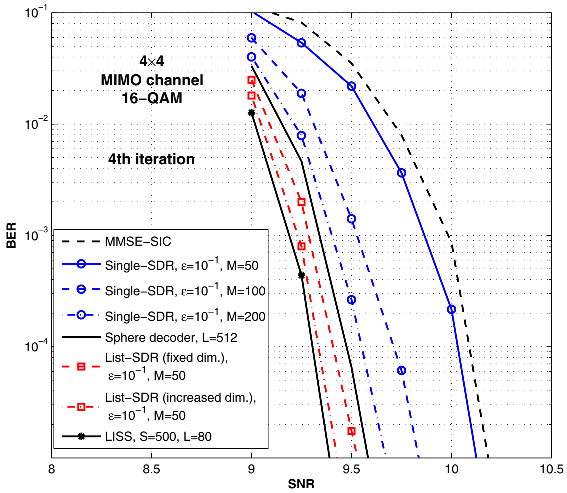

Fig. 2. Comparison of the BER performance.

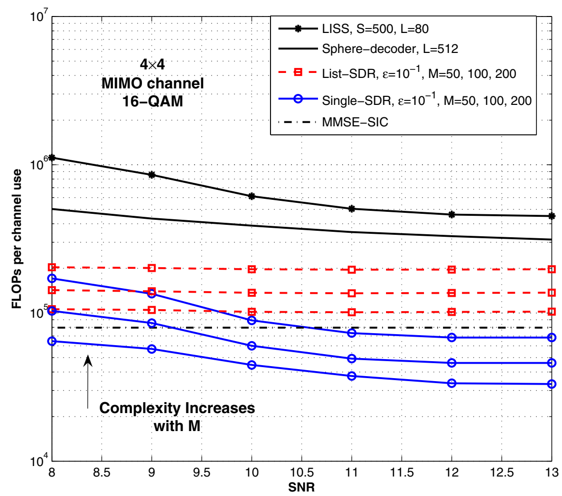

Fig. 3. Comparison of the average computational cost per channel use.

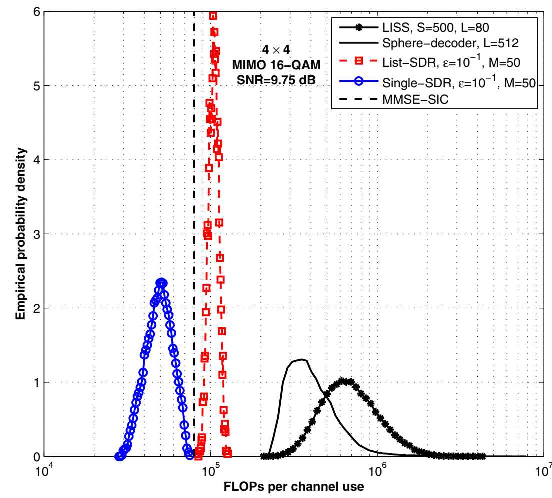

Fig. 4. Empirical probability density with logarithmic abscissa of the number of FLOPs per channel use.

that of the MMSE-SIC demodulator.

VII. Conclusion

Three computationally-efficient list-based soft demodulators have been developed for MIMO systems with 16-QAM signaling; two List-SDR demodulators and a Single-SDR demodulator. These schemes are based on the solution of different semidefinite programs (SDPs) and these solutions can be obtained in (low-order) polynomial time. The List-SDR scheme requires the solution of one SDP per iteration, and the Single-SDR scheme only requires the solution of one SDP per channel use. Simulation results illustrated that the computational advantages of the proposed demodulators are obtained without incurring a substantial degradation in performance.

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